1 Problem 1

Do the one-dimensional kinetic energy and momentum operators commute? If not, what operator does their commutator represent?

$$\hat{KE} \; = \; \frac{-\hbar^2}{2m}\,\frac{d^2}{dx^2} \qquad \hat{P} \; = \; -i\hbar\,\frac{d}{dx}$$

1.1 Solution

This question requires calculating the commutator of the operators given.

$$\begin{aligned}
\left[\hat{KE},\ \hat{P}\right] &= \left[\left(\frac{-\hbar^2}{2m}\,\frac{d^2}{dx^2}\right)\left(-i\,\hbar\,\frac{d}{dx}\right) - \left(-i\,\hbar\,\frac{d}{dx}\right)\left(\frac{-\hbar^2}{2m}\,\frac{d^2}{dx^2}\right)\right] \\
&= \frac{i\,\hbar^3}{2m}\left[\left(\frac{d^2}{dx^2}\right)\left(\frac{d}{dx}\right) - \left(\frac{d}{dx}\right)\left(\frac{d^2}{dx^2}\right)\right] \\
&= \frac{i\,\hbar^3}{2m}\left[\left(\frac{d^3}{dx^3}\right) - \left(\frac{d^3}{dx^3}\right)\right] \\
&= 0
\end{aligned}$$

The operators commute; all done.

2 Problem 2

Given the following wavefunction, describing some quantum particle, expanded in a basis of eigenfunctions of a "location" operator,

$$\Psi \; = \; C_{left}\,\psi_{left} \; + \; C_{right}\,\psi_{right} \; + \; C_{top}\,\psi_{top} \; + \; C_{bottom}\,\psi_{bottom}$$

NOTE:(the eigenfunction ψ_{left} is associated with an eigenvalue of "LEFT", and so on for the other eigenfunctions)

2a. What is the probability of observing a value of "TOP" (before any explicit measurement is made).

If we have a detector that measures the "LEFT" location, and a signal is detected there after a particle is projected towards the array of detectors:

2a. How would you write the expression for the wavefunction?

2b. What is the probability of observing a value of "LEFT"?

2c. If a measurement was made on the wavefunction 5 hours later, what would be the probability of measuring a value of "BOTTOM" (assume no interactions of the particle with external fields, particles, or other have occurred during this time)?

2.1 Solution

2a. C_{top}^2
2a. $\Psi \ = \ \psi_{left}$. **Note** $C_{left} \ = \ 1$
2b. 1
2c. 0

3 Problem 3

3a. What is the first excited state wavefunction for the one-dimensional harmonic oscillator (use a generic normalization constant for the current purposes)?

3b. Set up the equation for determining the average kinetic energy for this state (that of part 3a).
3c. What is the average value of x?

3.1 Solution

3a. first excited state: $n = 1$:
From page 11-2 in handbook

$$\psi_n(x) \ = \ A_n \ H_n(\frac{x}{\alpha}) \ exp\left(\frac{-x^2}{2\alpha^2}\right)$$

H_n is determined from Table 11.1 in Handbook for $n = 1$. Thus, the total wavefunction (with generic normalization constant is)

$$\psi_{n=1}(x) \ = \ 2 \ A_n \ \left(\frac{x}{\alpha}\right) \ exp\left(\frac{-x^2}{2\alpha^2}\right)$$

Note that in this case, the complex conjugate of the wavefunction is the same as the wavefunction since we are dealing with real functions (this is a special case, be careful for the general problems you may encounter in the future)

$$\psi^*_{n=1}(x) \;=\; 2\ A_n\ \left(\frac{x}{\alpha}\right)\ exp\left(\frac{-x^2}{2\alpha^2}\right)$$

3b. average kinetic energy:

$$\langle KE\rangle \;=\; \frac{\int_{-\infty}^{\infty}\ \psi^*(x)\ \hat{KE}\ \psi(x)\ dx}{\int_{-\infty}^{\infty}\ \psi^*(x)\ \psi(x)\ dx}$$

The kinetic energy operator is $\hat{KE} \;=\; \frac{-\hbar^2}{2m}\ \frac{d^2}{dx^2}$ Thus, using the above expressions for the wavefunction and its complex conjugate:

$$\langle KE\rangle \;=\; \frac{\int_{-\infty}^{\infty}\ \psi^*(x)\ \frac{-\hbar^2}{2m}\ \frac{d^2}{dx^2}\ \psi(x)\ dx}{\int_{-\infty}^{\infty}\ \psi^*(x)\ \psi(x)\ dx}$$

$$= \frac{\int_{-\infty}^{\infty}\ \left[2\ A_n\ \left(\frac{x}{\alpha}\right)\ exp\left(\frac{-x^2}{2\alpha^2}\right)\right]\ \left[\frac{-\hbar^2}{2m}\ \frac{d^2}{dx^2}\ \left(2\ A_n\ \left(\frac{x}{\alpha}\right)\ exp\left(\frac{-x^2}{2\alpha^2}\right)\right)\right]\ dx}{\int_{-\infty}^{\infty}\ \left[\ 2\ A_n\ \left(\frac{x}{\alpha}\right)\ exp\left(\frac{-x^2}{2\alpha^2}\right)\right]\ \left[2\ A_n\ \left(\frac{x}{\alpha}\right)\ exp\left(\frac{-x^2}{2\alpha^2}\right)\right]\ dx}$$

3c. Average value of x.

$$\langle x\rangle \;=\; \frac{\int_{-\infty}^{\infty}\ \psi^*(x)\ x\ \psi(x)\ dx}{\int_{-\infty}^{\infty}\ \psi^*(x)\ \psi(x)\ dx}$$

We will consider the numerator and denominator individually. The numerator is:

$$\int_{-\infty}^{\infty}\ \psi^*(x)\ x\ \psi(x) \;=\; \int_{-\infty}^{\infty}\ \frac{4\ A_n^2}{\alpha^2}\ x^3\ e^{-x^2/\alpha^2}\ dx \;=\; \frac{4\ A_n^2}{\alpha^2}\int_{-\infty}^{\infty}\ x^3\ e^{-x^2/\alpha^2}\ dx$$

Consider the symmetry of the integrand, $x^3\ e^{-x^2/\alpha^2}$. This is of **odd symmetry**. Thus, we can easily write down the answer to the integral as 0!

Now, let's just make sure that the denominator is not zero.

$$\begin{aligned}\int_{-\infty}^{\infty} \psi^*(x)\ \psi(x)\ dx &= \frac{4\ A_n^2}{\alpha^2} \int_{-\infty}^{\infty} x^2\ e^{-x^2/\alpha^2}\ dx \\ &= \frac{8\ A_n^2}{\alpha^2} \int_{0}^{\infty} x^2\ e^{-x^2/\alpha^2}\ dx\end{aligned}$$

The solution to the last integral is found in Table 2.3 (fifth integral on the left side). Thus, the denominator is not zero, and the average value of x is 0 (as we would expect intuitively).

4 Extra Credit: 10 points

For the n=2 stationary state of a 1-D particle in a box, what are the possible values of the momentum? What is the probability of each?
Consider the following helpful Euler relations:

$$\begin{aligned} e^{ikx} &= cos(kx)\ +\ i\ sin(kx) \\ e^{-ikx} &= cos(kx)\ -\ i\ sin(kx)\end{aligned}$$

4.1 Solution

For the $n = 2$ wavefunction for a 1-D particle in a box, the wavefunction is:

$$\psi(x)\ =\ \sqrt{\frac{2}{a}}\ sin\left(\frac{2\ \pi\ x}{a}\right)$$

We are asked for the values of momentum that would be observable for this particle. If the quantum mechanical operator for momentum were to operate on this wavefunction, we would see that the form of the wavefunction given above is **not** an eigenfunction of the momentum operator. Thus, we have no way of telling the values of momentum we would observe with the wavefunction as given above. We need to modify it; specifically, we need to rewrite it as **an expansion in the basis eigenfunctions of the momentum operator**.

To do this, we can use the Euler relations given. Specifically, we can subtract the two functions to obtain:

$$
\begin{aligned}
e^{-ikx} \ - \ e^{ikx} \quad &= \quad cos(kx) \ - \ i \ sin(kx) - cos(kx) \ - \ i \ sin(kx) \\
&= \ -2 \ i \ sin(kx)
\end{aligned}
$$

Thus, a form for the sin(kx) function in terms of the exponentials is:

$$
sin(kx) \ = \ \frac{-1}{2 \ i} \ e^{-ikx} \ + \ \frac{1}{2 \ i} \ e^{ikx}
$$

The exponentials are eigenfunctions of the momentum operator.

Returning to the 1-D particle in a box wavefunction for $n = 2$, we can by analogy:

$$
\sqrt{\frac{2}{a}} \ sin\left(\frac{2 \ \pi \ x}{a}\right) \ = \sqrt{\frac{2}{a}} \ \left(\frac{-1}{2 \ i} \ e^{\frac{-\ 2 \ i \ \pi \ x}{a}} \ + \ \frac{1}{2 \ i} \ e^{\frac{2 \ i \ \pi \ x}{a}}\right)
$$

Operating on the individual exponentials with the momentum operator, we obtain for the eigenvalues:

$$
\frac{2 \ \pi \ \hbar}{a} \ \ and \ \ \frac{-2 \ \pi \ \hbar}{a}
$$

Since the magnitude of the coefficients for the two eigenfunctions in the expansion are equal, their squares are equal. Since the probability of observing either wavefunction's observable (the eigenvalue) is equal to that of the other, the probability for each value of momentum is 0.5.