Hamiltonian Mechanics

The mechanics of Newton and Galileo focused on forces as the principal concept in describing change in a system. In this structure, the principal object is, for a given force field, to solve for the trajectories of particles. The method involved Newton's equation:

$$\mathbf{F} = m\mathbf{a}$$

where **F** is the force (which depends on space and can be temporal), *m* is the mass, and **a** is the acceleration.

In the latter part of the nineteenth century, the Irish physicist William Rowan Hamilton developed an alternative mathematical structure, which emphasized the energy as a significant determinant of the state of a mechanical system. The central element of this theoretical structure was the Hamiltonian function, a description of the total energy in terms of the momenta (**p**) and positions (**r**) of particles. It has the form

$$H(\mathbf{p},\mathbf{r}) = E = T(\mathbf{p}) + V(\mathbf{r})$$

where $T(\mathbf{p})$ is the kinetic energy expressed in terms of the momentum, and $V(\mathbf{r})$ is the potential expressed in terms of the positions of all particles in the system.

Knowing the Hamiltonian function for a conservative system (one having constant energy), one obtains the equations of motion by the use of Hamilton's canonical equations:

$$\frac{\partial H}{\partial r_i} = -\frac{dp_i}{dt}$$
$$\frac{\partial H}{\partial p_i} = \frac{dr_i}{dt}$$

where p_i and r_i are conjugate variables such as (p_x, x). The first of the canonical equations is functionally equivalent to Newton's equation given above. The second is a relationship between a component of the momentum and a component of the velocity. Solving Hamiltonian's canonical equations is equivalent to solving Newton's equations of motion, but the connection of the state (trajectory) to the energy is obvious in Hamilton's formulation.

For systems of multiple particles, it is easy to form the Hamiltonian by counting all sources of kinetic energy and all sources of potential energy. Consider a system of two particles (with labels 1 and 2), which interact through a Coulomb potential. The Hamiltonian function for such a system is:

$$H = \frac{\mathbf{p}_1 \cdot \mathbf{p}_1}{2m_1} + \frac{\mathbf{p}_2 \cdot \mathbf{p}_2}{2m_2} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\mathbf{r}_2 - \mathbf{r}_1|}$$