MATH 616-010 Modeling in Applied Mathematics
Prof. D. A. Edwards Oct. 22, 2023

Stefan Problem

The *error function* is defined by

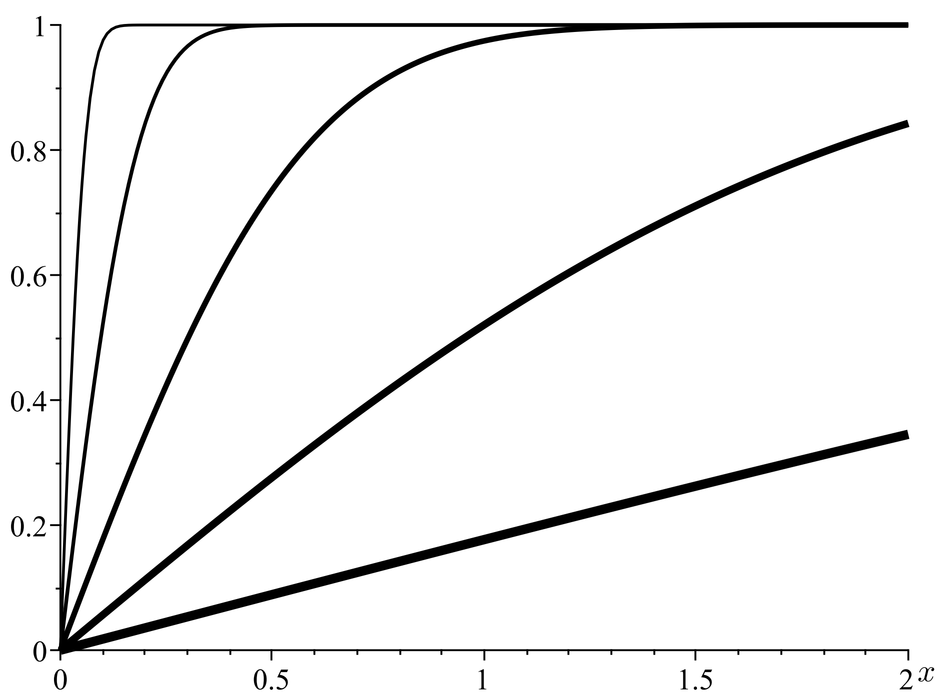
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz.$$

The coefficient in front of the integral normalizes $\operatorname{erf}(\infty) = 1$. Clearly $\operatorname{erf} x$ is odd in x.

The error function arises often in probability problems. Also, because of the relationship between the random walk and diffusion processes, it occurs often in diffusion problems. In particular, in the example in class, we found that the solution to the Stefan problem was related to

$$\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right).$$

This function is plotted below for various values of t. Note that increasing t doesn't change the *shape* of the curve; it just changes the *scale*.

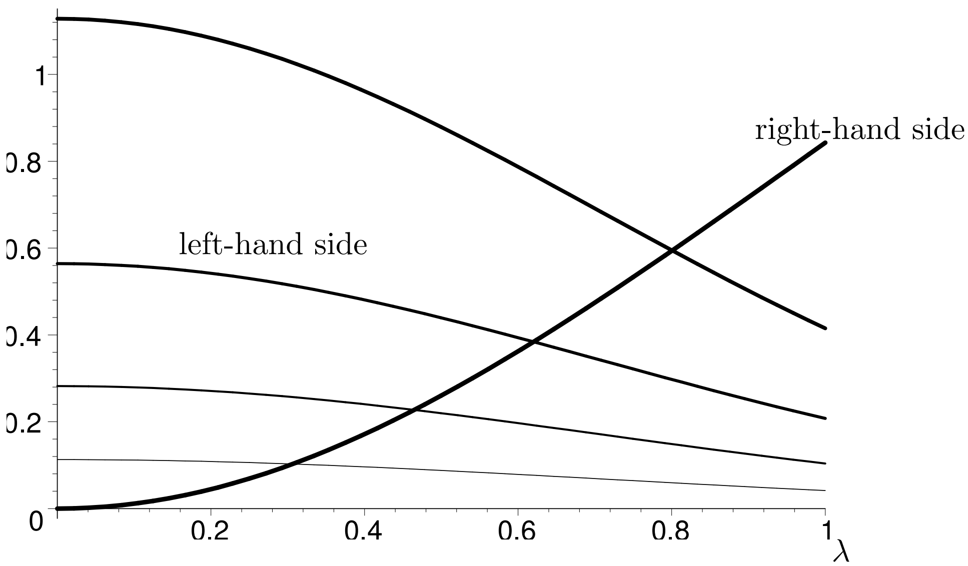


$\operatorname{erf}(x/2\sqrt{t})$ *vs.* x. In increasing order of thickness: $t = 0.001$, 0.01, 0.1, 1, 10.

In class we determined that the solution of the Stefan problem was given by

$$T(x,t) = 1 - \frac{1}{\operatorname{erf}\lambda}\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right), \tag{1a}$$

$$\frac{\mathrm{St}e^{-\lambda^2}}{\sqrt{\pi}} = \lambda \operatorname{erf}\lambda. \tag{1b}$$



Left- and right-hand sides of (1b) *vs.* λ. In increasing order of thickness: St $= 0.2,\ 0.5,\ 1,\ 2$.