Review

First-Order Equations

First-order equations come in two flavors that we can easily solve. First, a *separable equation* is one where the dependent and independent variables may be separated so that the equation may be integrated:

$$\begin{aligned} \dot{y} &= ty^3 \\ \frac{dy}{y^3} &= t\,dt \\ -\frac{1}{2y^2} &= \frac{t^2}{2} + C. \end{aligned}$$

Next, for a linear equation

$$\dot{y} + p(t)y = g(t),$$

one may multiply by an *integrating factor* $\mu = \exp(\int p\,dt)$ to obtain an integrable right-hand side, as in this example:

$$\begin{aligned} \dot{y} - 2ty &= \frac{2}{\sqrt{\pi}} e^{t^2}, \qquad p(t) = -2t, \quad \mu(t) = e^{-t^2} \\ e^{-t^2}\dot{y} - 2te^{-t^2}y &= \frac{2}{\sqrt{\pi}} \\ \frac{d}{dt}\left(e^{-t^2}y\right) &= \frac{2}{\sqrt{\pi}} \\ y &= e^{t^2}\left(\frac{2t}{\sqrt{\pi}} + C\right). \end{aligned}$$

Second-Order Equations

Similarly, second-order equations come in two flavors that we can easily solve. For a linear equation with constant coefficients, we may substitute $y = e^{\lambda t}$ to obtain a quadratic equation for λ:

$$a\ddot{y} + b\dot{y} + cy = 0 \qquad \Longrightarrow \qquad a\lambda^2 + b\lambda + c = 0.$$

Each λ gives rise to a solution except in the case of repeated roots.

Next, for an *Euler equation*

$$at^2\ddot{y} + bt\dot{y} + cy = 0,$$

substituting $y = t^\lambda$ again yields a quadratic equation for λ:

$$a\lambda(\lambda - 1) + b\lambda + c = 0.$$

Again, each λ gives rise to a solution except in the case of repeated roots.

If the equation has a nonzero right-hand side, there are two methods to find the particular solution. If we have constant coefficients, one may use the *method of undetermined coefficients*. In this method, one tries an unknown multiple of the right-hand side and then tries to find the coefficient that satisfies the equation. For instance:

$$y_{\mathrm{p}} = Ae^{2t} \qquad \Longrightarrow \qquad \begin{aligned} \ddot{y} - y &= e^{2t} \\ 4A - A &= 1 \\ y_{\mathrm{p}} &= \frac{e^{2t}}{3}. \end{aligned}$$

Alternatively, one may use the *variation of parameters* formula:

$$y_{\mathrm{p}} = -y_1 \int \frac{y_2 g}{W}\,dt + y_2 \int \frac{y_1 g}{W}\,dt,$$

where $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous equation, g is the right-hand side, and W is the *Wronskian* $y_1\dot{y}_2 - \dot{y}_1 y_2$. In the case above, substituting $e^{\lambda t}$ into the homogeneous equation yields $\lambda = \pm 1$, so let $y_1 = e^t$, $y_2 = e^{-t}$. Then $W = -2$ and the formula becomes

$$\begin{aligned} y_{\mathrm{p}} &= -e^t \int \frac{e^{-t}e^{2t}}{(-2)}\,dt + e^{-t} \int \frac{e^t e^{2t}}{(-2)}\,dt, \\ &= e^t \left(\frac{e^t}{2}\right) - e^{-t}\left(\frac{e^{3t}}{6}\right) = \frac{e^{2t}}{3}, \end{aligned}$$

so the answers match.

Laplace Transforms

Another way to solve ODEs with constant coefficients is with *Laplace transforms.* If we define $\mathcal{L}[y] \equiv \hat{y}$ to be the Laplace transform of $y(t)$, then $\hat{y}$ satisfies the following useful properties:

$$\mathcal{L}[\dot{y}] = s\hat{y} - y(0).$$

This formula (and its extensions to multiple derivatives) allows us to transform ODEs, solve for $\hat{y}$, and then use tables or shifting formulas to find y.

Another useful property is that

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right] = \hat{f}\hat{g}.$$

This *convolution property* is useful for solving ODEs with general right-hand sides, as well as solving integral equations.

For more details about ODEs, a good place to start is the book *Elementary Differential Equations and Boundary Value Problems* by Boyce, DiPrima, and Meade, which we use on campus in our undergraduate courses.

Linear Algebra

Recall that if $A \in \mathcal{R}^{n\times n}$ has a complex eigenvalue λ such that $A\mathbf{z} = \lambda\mathbf{z}$, then $A\bar{\mathbf{z}} = \bar{\lambda}\bar{\mathbf{z}}$. Also, if we denote an eigenvalue for A as $\lambda(A)$, then $[\lambda(A)]^n = \lambda(A^n)$ for any integer n.

Probability

The *Poisson process* is often used to measure random discrete events (such as waiting times, encounters, etc.). It is given by

$$P(t = n) = \frac{e^{-\mu}\mu^n}{n!}, \quad \mu > 0,$$

where $n \geq 0$ is an integer. Note that the larger n, the smaller the probability (since $n!$ grows faster than μ^n for any μ). Also, the mean of the distribution is given by μ.

Recall that the mean of any random variable X, usually given the variable name μ, may be denoted as

$$\mu = \langle X \rangle.$$

Moreover, the variance, usually given the variable σ^2, may be denoted as

$$\sigma^2 = \langle (X - \langle X \rangle)^2 \rangle.$$

If we repeat the same sort of trial over and over, the variables X_i are said to be *independent* and *identically distributed* (iid). In that case, we have

$$\left\langle \sum_{i=1}^{N} X_i \right\rangle = N\langle X \rangle, \qquad \operatorname{Var}\left\langle \sum_{i=1}^{N} X_i \right\rangle = N \operatorname{Var} X.$$

Calculus

Leibniz's Rule is an extension of the Fundamental Theorem of Calculus used for more complicated integrals:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)\,d\tau = \frac{db}{dt} f(t, b(t)) - \frac{da}{dt} f(t, a(t)) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(t,\tau)\,d\tau.$$

(Note that this result reduces to the Fundamental Theorem of Calculus when a and f are independent of t, and $b(t) = t$.)

Suppose that we have a function $T(x,y)$, and we want to know how it changes as we move through the xy-plane. For instance, suppose T is the temperature of a plate, and we want to know how the temperature would change as a thermometer traces a path $(x(t), y(t))$ on it. So the desired temperature can be written as $T(x(t), y(t))$. Then

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

by the Chain Rule.

Gauss' Theorem says that if V is a volume bounded by the surface ∂V, then

$$\int_V f\, dV = \int_{\partial V} \nabla f \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the *outward-pointing* normal to the surface ∂S.

The divergence of a tensor (matrix) A is given by taking the dot product of the gradient vector with each *column*:

$$\nabla \cdot A = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{23} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \sum_{i=1}^{3} \left(\frac{\partial a_{i1}}{\partial x_i}, \frac{\partial a_{i2}}{\partial x_i}, \frac{\partial a_{i3}}{\partial x_i}\right).$$