MATH 616-010 | Modeling in Applied Mathematics
Prof. D. A. Edwards | Nov. 10, 2023

Conservation of Momentum

In its purest form, conservation of linear momentum is given by

$$\frac{D}{Dt}\int_{V(t)} \rho\mathbf{u}\,dV = \int_{V(t)} \rho\mathbf{f}\,dV + \int_{S(t)} \mathbf{t}\,dS \tag{1}$$

Then using the Reynolds Transport Theorem

$$\frac{D}{Dt}\int_{V(t)} g\,dV = \int_{V(t)} \frac{\partial g}{\partial t} + \nabla\cdot(g\mathbf{u})\,dV \tag{2}$$

in (1), we obtain

$$\int_{V(t)} \frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}^T) - \rho\mathbf{f}\,dV = \int_{S(t)} \mathbf{t}\,dS. \tag{3}$$

Using the Principle of Stress Equilibrium

$$\lim_{L\to 0}\frac{1}{L^2}\int_{S(t)} \mathbf{t}\,dS = \mathbf{0} \tag{4}$$

and Newton's Third Law

$$\mathbf{t}(\mathbf{n}) = -\mathbf{t}(-\mathbf{n}), \tag{5}$$

we found that $\mathbf{t} = T\cdot\mathbf{n}$, where T is the stress tensor. So (3) may be rewritten in the following differential form:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}^T) - \rho\mathbf{f} - \nabla\cdot T = \mathbf{0}. \tag{6}$$

Rewriting in terms of the material derivative, ignoring any body forces but gravity, and using conservation of mass, we obtain

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} + \nabla\cdot T. \tag{7}$$

Then rewriting $T = -pI + \tau$, we have

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} - \nabla p + \nabla\cdot\tau. \tag{8}$$

Lastly, for a Newtonian fluid we take $\nabla\cdot\tau = \mu\nabla^2\mathbf{u}$, which yields the final expression

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u}. \tag{9}$$