Hamiltonian Systems

In class, we discussed the solutions of the following system:

$$\dot{x} = \frac{\partial H}{\partial y},$$
$$\dot{y} = -\frac{\partial H}{\partial x},$$

and noted that H, the *Hamiltonian* function for the system, was constant along trajectories.

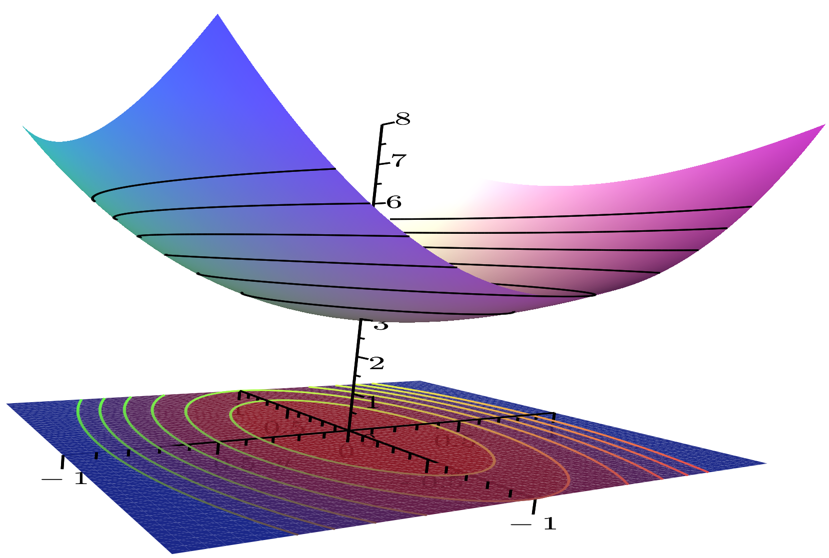
First we consider the case where

$$\begin{aligned} \dot{x} &= 2y \\ \dot{y} &= -8x \end{aligned} \qquad \Longrightarrow \qquad H = 4x^2 + y^2 + C.$$

The only fixed point is at the origin, where the relevant matrix is

$$A = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \lambda = \pm 4i.$$

Therefore, the origin is a center.



$H = 4x^2 + y^2 + 3$ with contour lines and level curves.

This figure shows the Hamiltonian function with contours in 3-D as well as the level curves projected onto the xy-plane to show the trajectories.

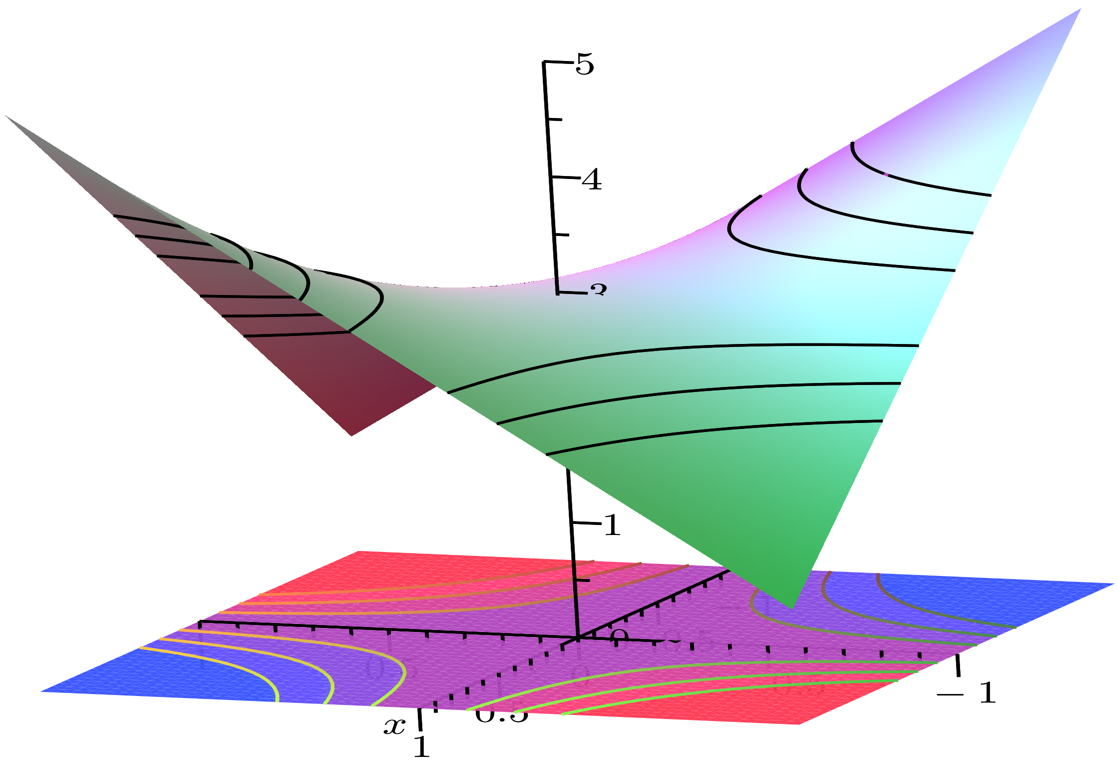
Next we consider the case where

$$\begin{aligned} \dot{x} &= 2x \\ \dot{y} &= -2y \end{aligned} \qquad \Longrightarrow \qquad H = 2xy + C.$$

The only fixed point is at the origin, where the relevant matrix is

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \qquad \Longrightarrow \qquad \lambda = \pm 2.$$

Therefore, the origin is a saddle point.



$H = 2xy + 3$ with contour lines and level curves.

This figure shows the Hamiltonian function with contours in 3-D as well as the level curves projected onto the xy-plane to show the trajectories. Note the saddle shape of the surface near the origin; this is the motivation for the term *saddle point*.