Envelopes

For the problem

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0, \qquad \rho(x,0) = -\tanh x,$$

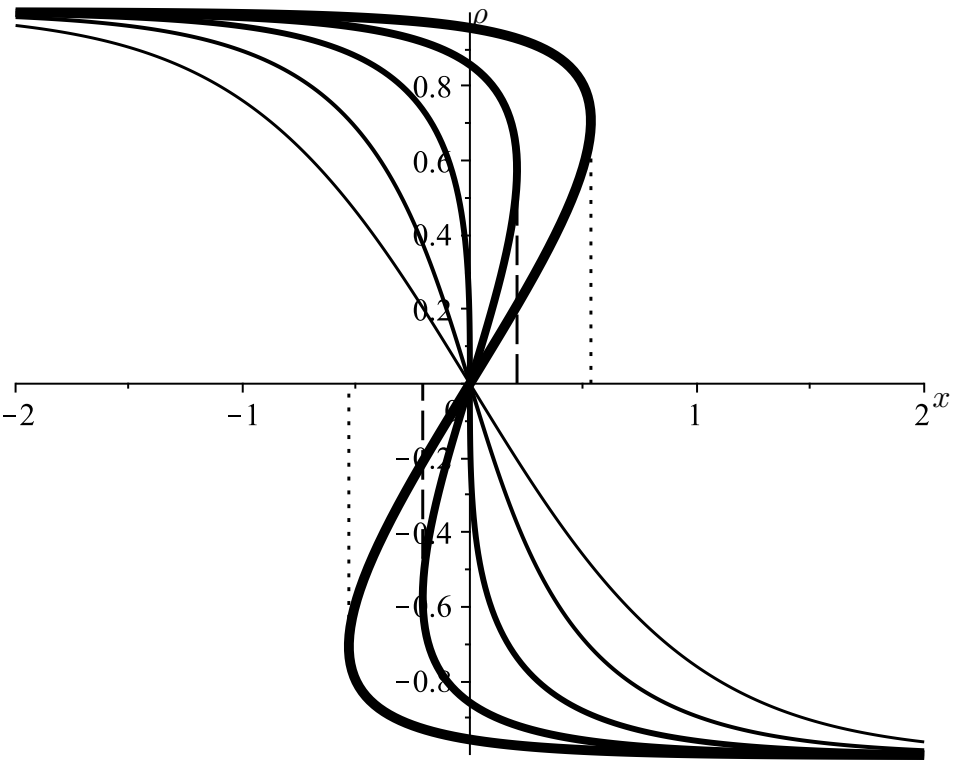
we determined that the solution is given by

$$\rho = -\tanh \xi \tag{1a}$$

along characteristics given by

$$x = \xi - (\tanh \xi) t. \tag{1b}$$

We also determined that $t_{\mathrm{B}} = 1$.

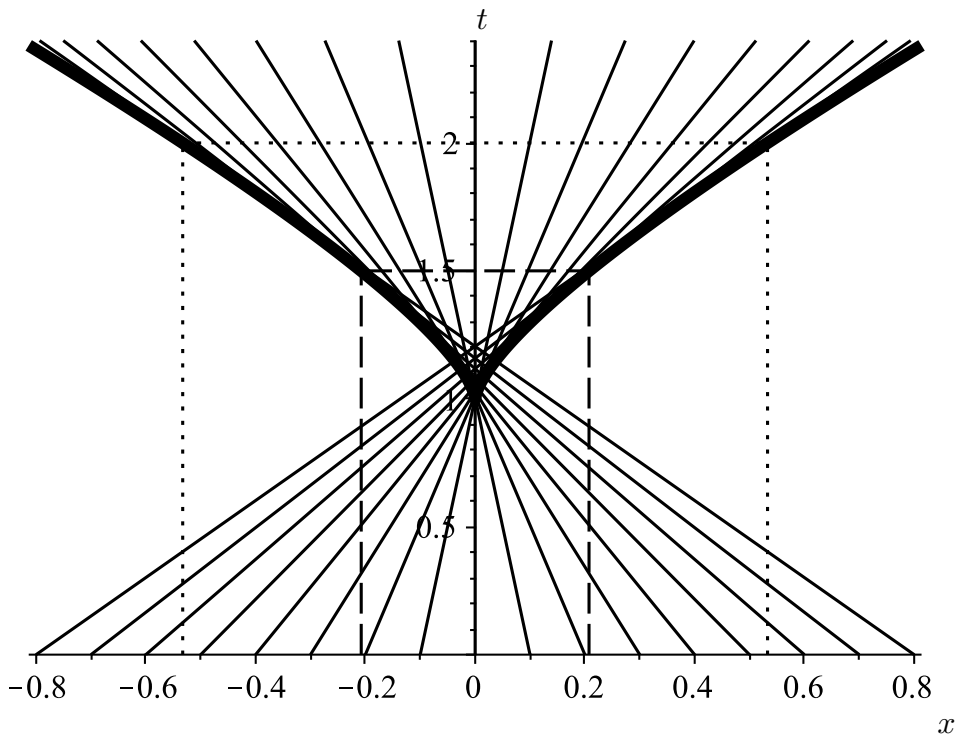


$\rho(x,t)$ *vs.* x as given by (1) with $t = 0$, 0.5, 1, 1.5, 2 (in increasing order of thickness). Note the single vertical tangent at $t_{\mathrm{B}} = 1$. The dashed and dotted lines show the positions of the vertical tangents at $t = 1.5$ and $t = 2$. Note these will correspond to the envelope.

We also determined that the envelope of intersecting characteristics is given by (1b) and

$$0 = 1 - (\operatorname{sech}^2 \xi)t. \tag{2}$$

Below are plotted characteristics given by (2) for $\xi = -0.8$ through $\xi = 0.8$ incremented by 0.1, as well as the envelope (thick line). Note that the envelope has a cusp at t_{B}.



Characteristics given by (1b) (thin lines) and envelope given by (1b) and (2) (thick line). The dashed and dotted lines illustrate the position of the envelope at $t = 1.5$ and $t = 2$, for comparison with the diagram on reverse.