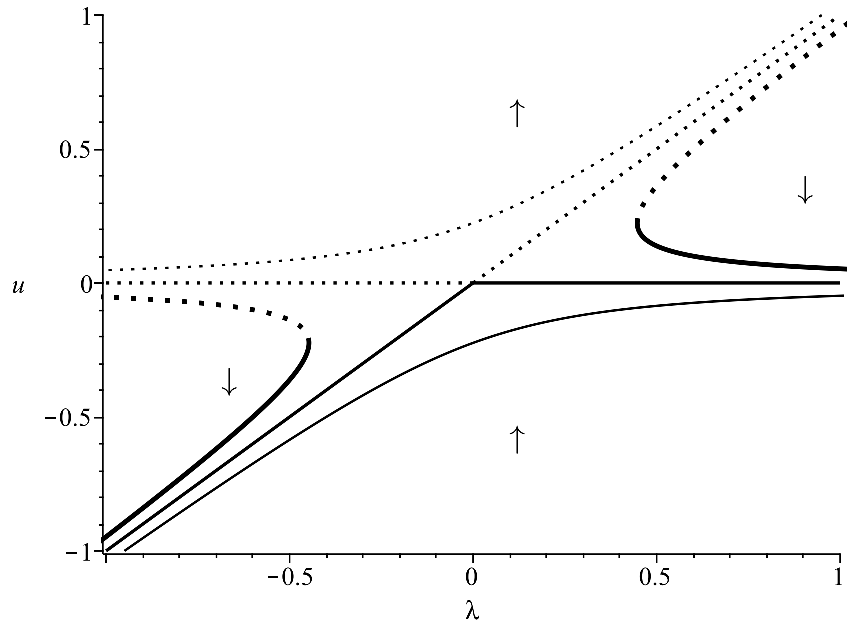
Transcritical Bifurcation

We examine the following canonical form of the perturbed transcritical bifurcation:

$$\frac{du}{dt} = u(u - \lambda) + \epsilon. \tag{1}$$

Here is a graph of the steady states of (1).



Steady states of (1) for $\epsilon = -0.05$, 0, and 0.05 (in increasing order of thickness).
Dotted curves: unstable. Solid curves: stable.

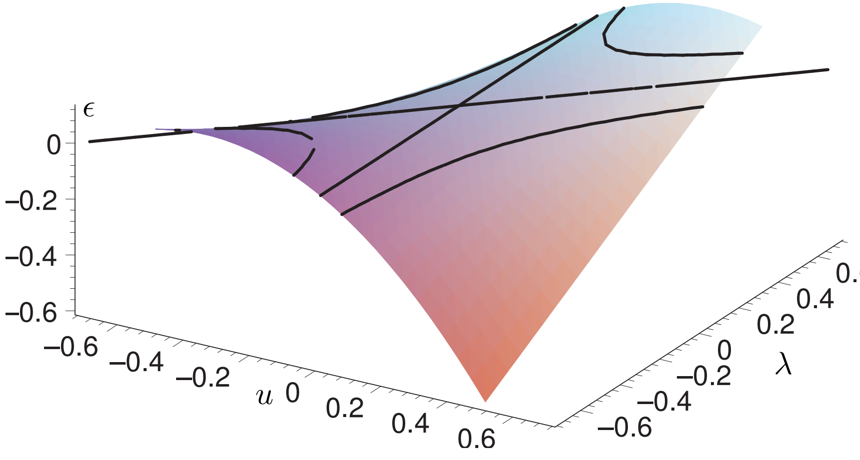
Remarks

1. The flow arrows for u do not change from the $\epsilon = 0$ case to the $\epsilon \neq 0$ case.
2. For $\epsilon < 0$ there are always two solutions and no folds.
3. For $\epsilon > 0$ there is not a solution for all values of λ. In particular, in class we showed that the fold in λ-ϵ space that divided between two solutions and no solution was given by

$$\lambda^2 = 4\epsilon. \tag{2}$$

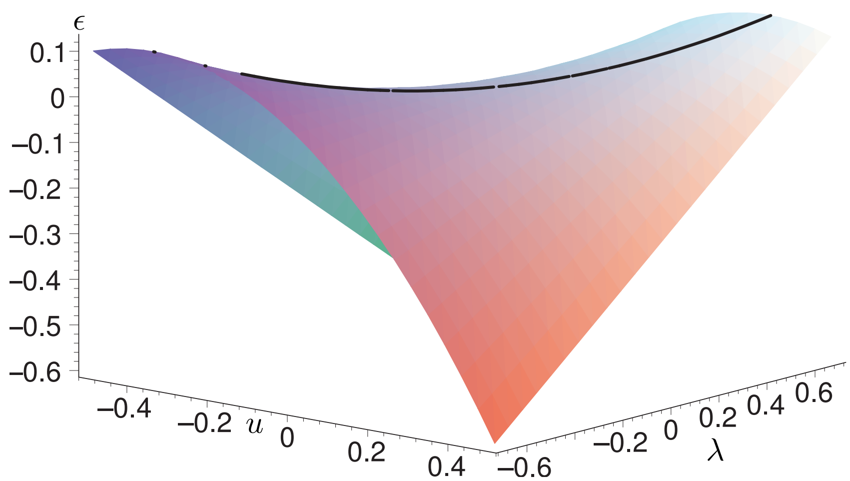
Thus, in this case, we have two folds which occur in λ-u space.

Here is a graph of the steady states of (1), with ϵ corresponding to the z-axis. The three curves on the previous page now appears as level curves.



Steady states of (1) with $\epsilon = 0, \pm 0.05$ labeled.

Here is a graph of the steady states of (1) *vs.* ϵ and λ, as well as the fold curve given by (2). Note that for (λ, ϵ) pairs above this curve, there are no u-intersections (and hence no solutions). Also, for (λ, ϵ) on this curve there is exactly one solution (since this curve forms the top of the saddle), and for (λ, ϵ) below this curve there are two solutions.



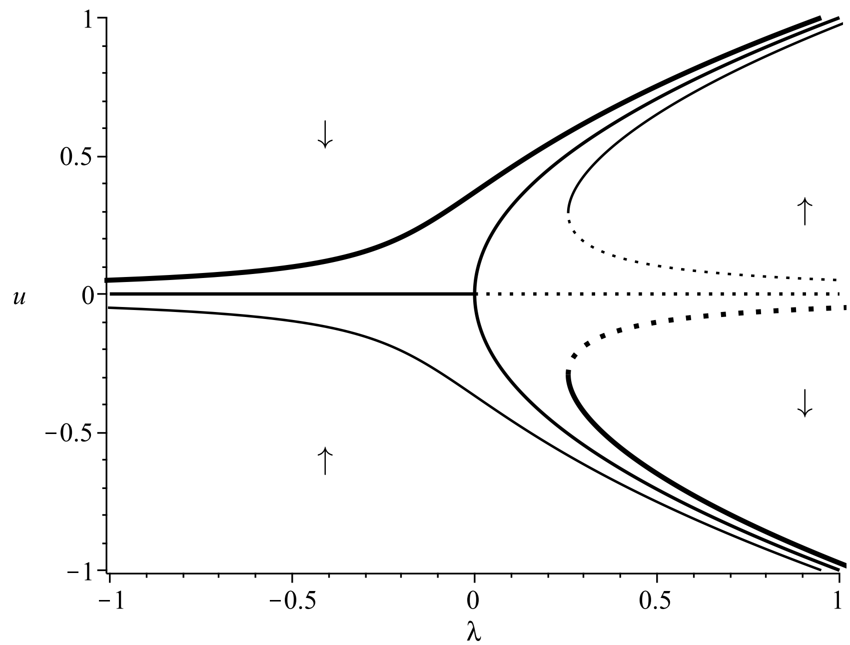
Steady states of (1) as well as fold curve (2).

Pitchfork Bifurcation

We examine the following canonical form of the perturbed pitchfork bifurcation:

$$\frac{du}{dt} = u(\lambda - u^2) + \epsilon. \tag{3}$$

Here is a graph of the steady states of (3).



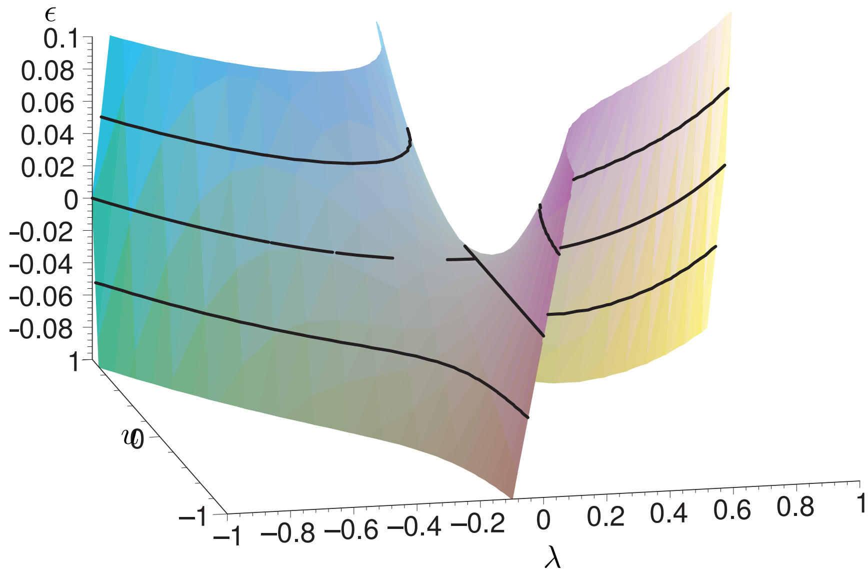
Steady states of (3) for $\epsilon = -0.05$, 0, and 0.05 (in increasing order of thickness).
Dotted curves: unstable. Solid curves: stable.

Remarks

1. The flow arrows for u do not change from the $\epsilon = 0$ case to the $\epsilon \neq 0$ case.
2. If $\epsilon < 0$, the top two branches coalesce form a fold in λ-u space; if $\epsilon > 0$ the bottom two branches do.
3. Depending on the value of ϵ, there may be one or three solutions for u. In particular, in class we showed that the fold in λ-ϵ space that divided between two solutions and no solution was given by

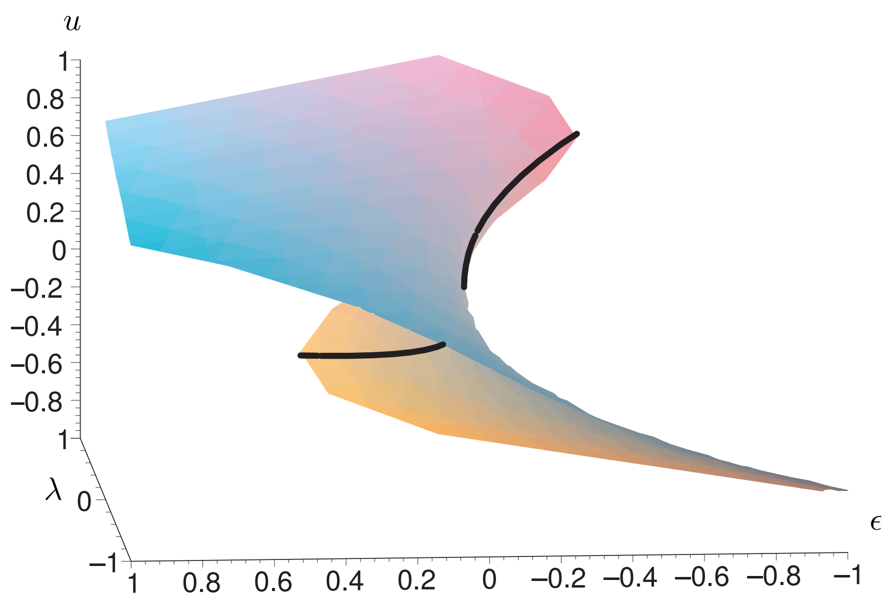
$$\lambda^3 = \frac{27\epsilon^2}{4}. \tag{4}$$

Here is a graph of the steady states of (3), with ϵ corresponding to the z-axis. The three curves on the previous page now appears as level curves.



Steady states of (3) with $\epsilon = 0, \pm 0.05$ labeled.

Here is a graph of the steady states of (3) *vs.* ϵ and λ, as well as the fold curve given by (4). Note that as we move into the plane of the paper (larger λ), we have three solutions, while we move out of the plane of the paper, we have only one.



Steady states of (3) as well as fold curve (4).