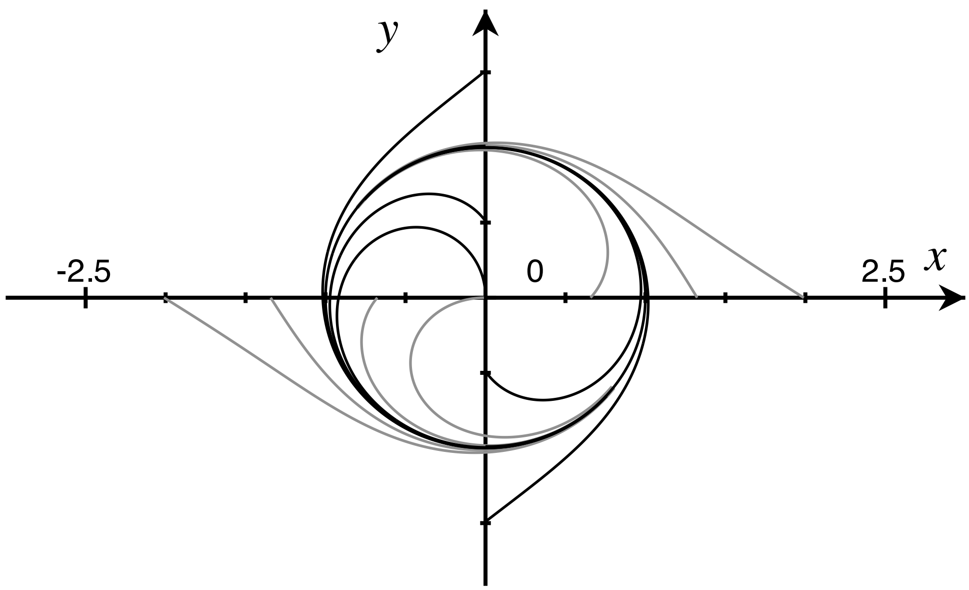
Action-Angle Coordinates

In class, we discussed the solutions of the following system in *action-angle* coordinates:

$$\dot{r} = f(r),$$
$$\dot{\theta} = \omega.$$

We noted that the sign of ω determined whether the trajectories spun clockwise or counterclockwise, while the existence and stability of limit cycles could be determined from the $\dot{r}$ equation.

In all the figures that follow, $\omega > 0$, so we have counterclockwise rotation.

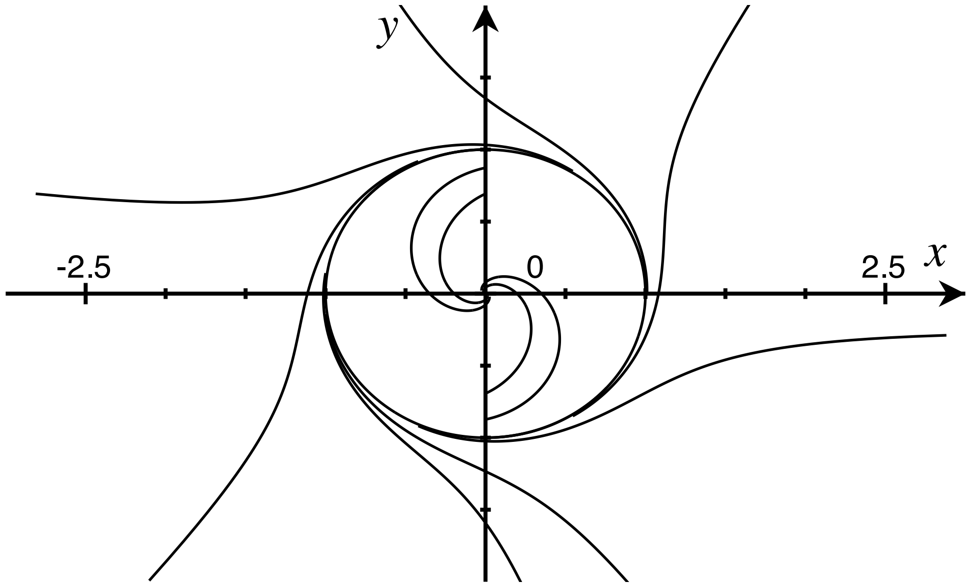


$f(r) = 1 - r^2$.

This figure shows the case where $f(r) = 1 - r^2$. Here there is a stable limit cycle at $r = 1$. We found that the solution is given by

$$r(t) = \frac{C - e^{-2t}}{C + e^{-2t}},$$

where C is a constant related to the initial condition. Note that $r(\infty) = 1$ for all C, which confirms that $r = 1$ is a stable limit cycle.

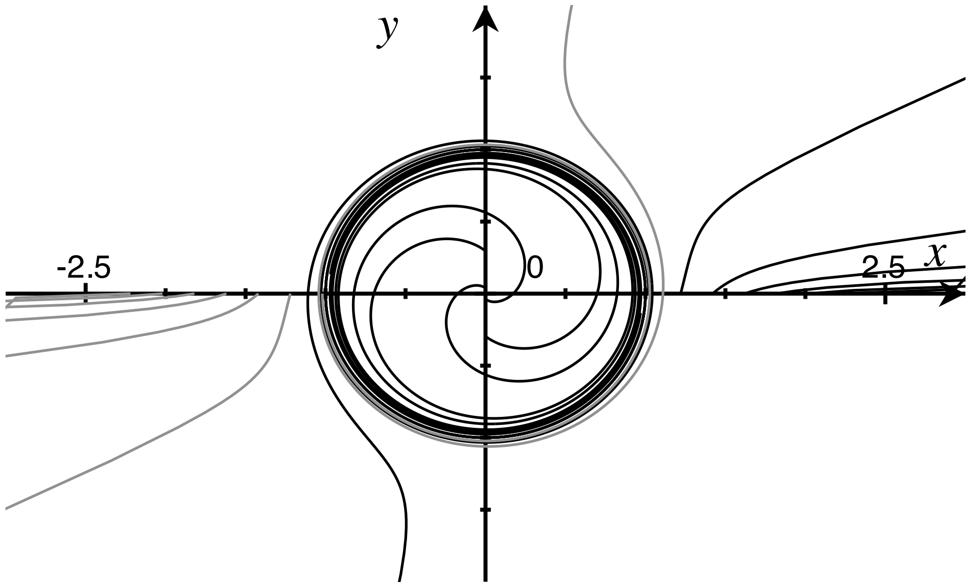


$f(r) = r(r^2 - 1)$.

This figure shows the case where $f(r) = r(r^2 - 1)$. Here there is an unstable limit cycle at $r = 1$; this is essentially the opposite of the Hopf bifurcation diagram. We found that the solution was

$$r(t) = (1 + Ce^{2t})^{-1/2},$$

where C is a constant related to the initial condition. Note that $r(-\infty) = 1$, so the limit cycle at $r = 1$ is unstable.



$f(r) = r(r^2 - 1)^2$.

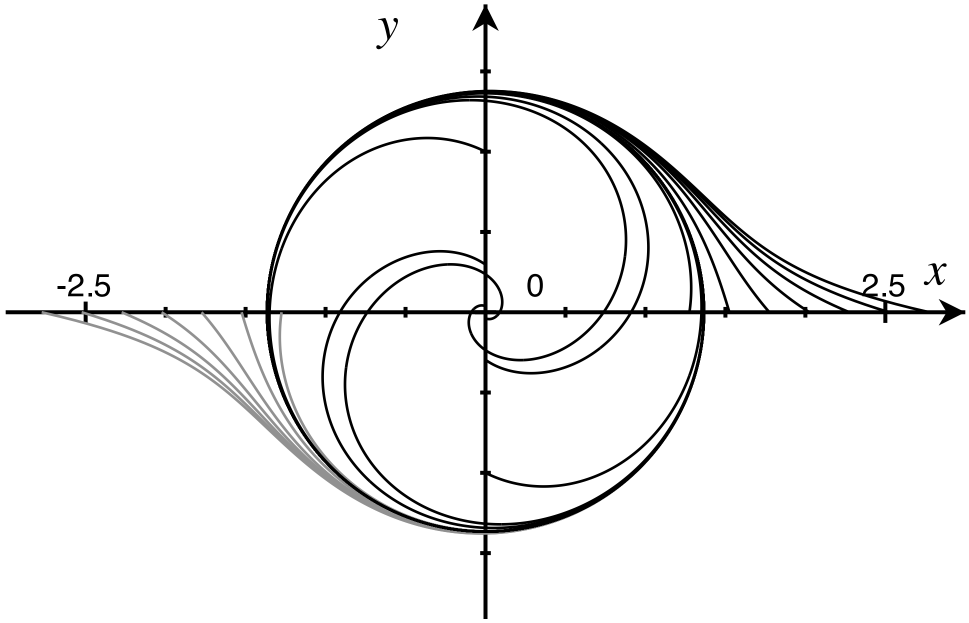
This figure shows the case where $f(r) = r(r^2 - 1)^2$. Here there is a semistable limit cycle at $r = 1$. Note the extra windings indicating a very slow approach/departure from $r = 1$. We found the solution in implicit form:

$$r^2 \exp\left(-2t - \frac{1}{r^2 - 1}\right) = C(r^2 - 1),$$

where $\operatorname{sgn}(r - 1) = \operatorname{sgn}(C)$ for all time.

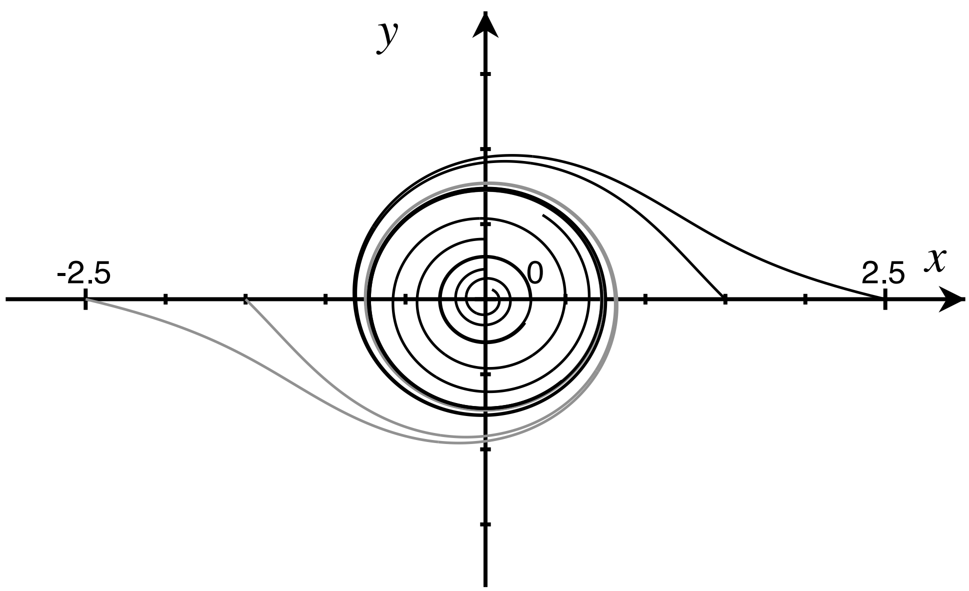
The next three diagrams concern the case where

$$f(r) = -r(r^2 - r + \lambda).$$



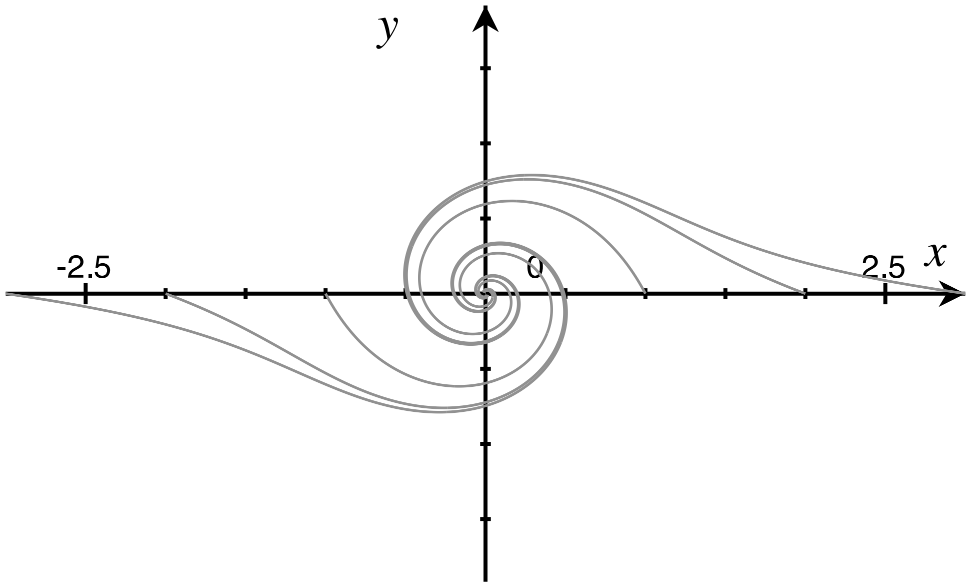
$\lambda = -1/2.$

In class we showed that if $\lambda < 0$, the origin is an unstable spiral and there is one stable limit cycle. This case is shown above.



$\lambda = 0.2.$

In class we showed that if $0 < \lambda < 1/4$, the origin is a stable spiral, then moving outward one finds one unstable and one stable limit cycle. This case is shown above.



$\lambda = 1/2.$

In class we showed that if $\lambda > 1/4$, the origin is stable and there are no limit cycles. This case is shown above.