Single-Species Differential Equations

In class, we obtained the following dimensionless model for the population N of the spruce budworm:

$$\dot{N} = rN\left(1 - \frac{N}{q}\right) - P(N), \qquad P(N) = \frac{N^2}{1+N^2}.$$

Below is shown a graph of the predation term. Note that it starts off small, then saturates at the value 1 as $N \to \infty$.

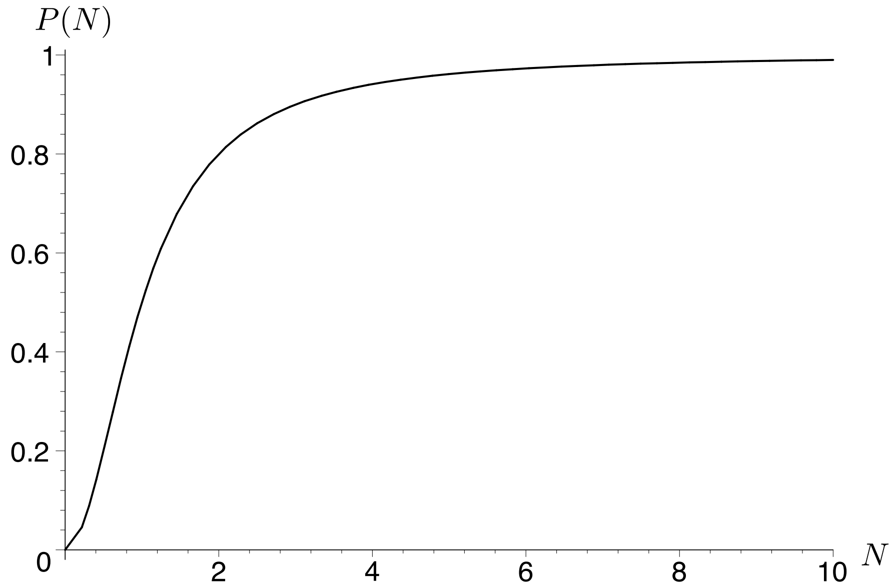

Figure 1. Dimensionless predation term.

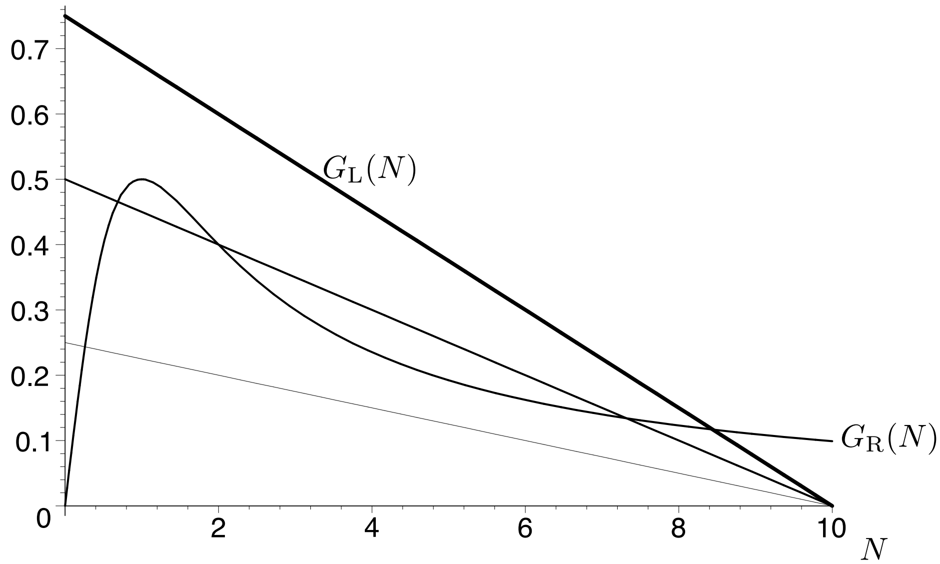

Figure 2. $G_{\mathrm{L}}(N)$ and $G_{\mathrm{R}}(N)$ for $q = 10$. N values of the intersections correspond to the steady states. In increasing order of thickness: $r = 1/4$, $r = 1/2$, and $r = 3/4$.

We found that in addition to the case $N = 0$, the fixed points were given by the solutions of

$$G_{\mathrm{L}}(N) \equiv r\left(1 - \frac{N}{q}\right) = \frac{N}{1 + N^2} \equiv G_{\mathrm{R}}(N). \tag{1}$$

A graph of the G functions is shown above. Note that as we increase r for a given value of q we have one, three, and then again one root.

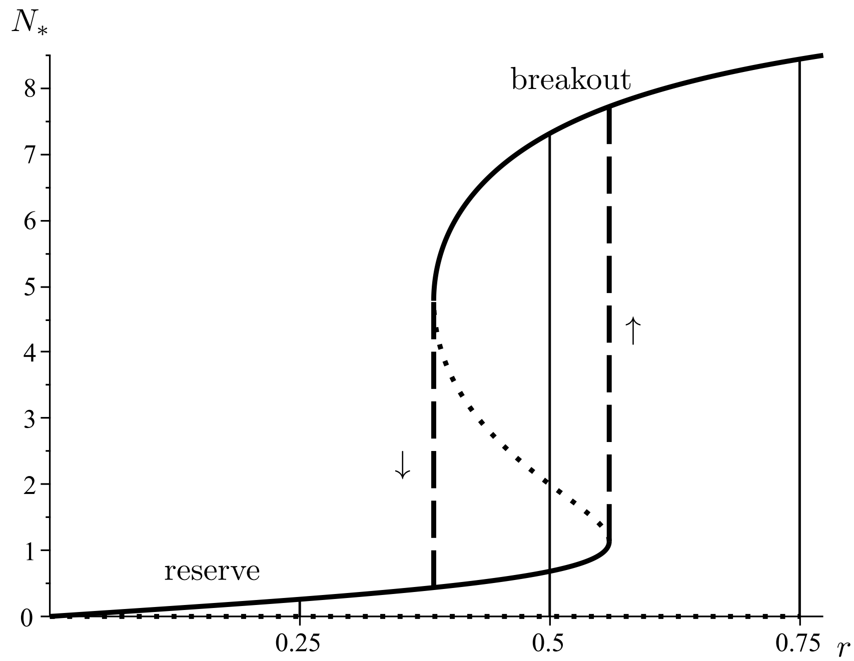

Figure 3. Hysteresis curve for $q = 10$. Solid curve: stable solutions. Dotted curve: unstable solutions. Dashed lines: actual transitions as r changes. Vertical lines: r values plotted in Fig. 2.

This effect is shown in the hysteresis curve for the same value of q, as shown above. [The stability of each solution is found by comparing the slopes of $G_{\rm L}(N)$ and $G_{\rm R}(N)$ at the intersection points.] The solution may be forced to the reserve population by reducing r (shown above) or reducing q (shown below).

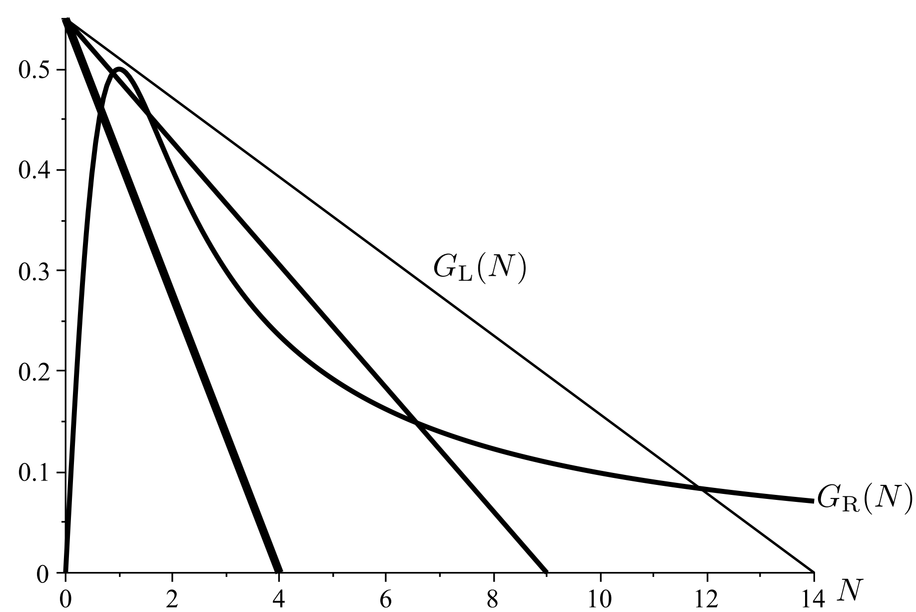

Figure 4. $G_{\rm L}(N)$ and $G_{\rm R}(N)$ for $r = 0.55$. N values of the intersections correspond to the steady states. In increasing order of thickness: $q = 14$, $q = 9$, and $q = 4$.