Homework Set 2 Solutions

1. Suppose that in the spruce budworm model we replace the predation term by

$$\tilde{P}(\tilde{N}) = \frac{B}{2}\left[1 + \tanh\left(\frac{\tilde{N} - A}{N_{\mathrm{w}}}\right)\right],$$

where $N_{\mathrm{w}} \ll A$ models the width of the transition region from no predation to full predation.

(a) (5 points) Use the same scalings as those given in class to scale the evolution equation that results.

Solution. The new equation is

$$\frac{d\tilde{N}}{d\tilde{t}} = R\tilde{N}\left(1 - \frac{\tilde{N}}{K}\right) - \frac{B}{2}\left[1 + \tanh\left(\frac{\tilde{N} - A}{N_{\mathrm{w}}}\right)\right].$$

Letting

$$\tilde{N} = NA, \qquad \tilde{t} = \frac{At}{B}, \qquad R = \frac{Br}{A}, \qquad K = Aq,$$

we have

$$\begin{aligned} B\dot{N} &= RAN\left(1 - \frac{N}{q}\right) - \frac{B}{2}\left[1 + \tanh\left(\frac{N-1}{N_{\mathrm{w}}/A}\right)\right] \\ \dot{N} &= rN\left(1 - \frac{N}{q}\right) - \frac{1}{2}\left[1 + \tanh\left(\frac{N-1}{N_{\mathrm{w}}/A}\right)\right]. \end{aligned} \tag{A}$$

Now let $N_{\mathrm{w}}/A = \epsilon$, where $0 < \epsilon \ll 1$.

(b) (2 points) As we take the limit $\epsilon \to 0$, what values does our dimensionless predation term take on?

Solution. Our dimensionless predation term is now

$$P(N) = \frac{1}{2}\left[1 + \tanh\left(\frac{N-1}{\epsilon}\right)\right],$$

which takes on the values

$$P(N) = \begin{cases} 0, & N < 1, \\ 1, & N > 1, \end{cases} \qquad \epsilon \to 0.$$

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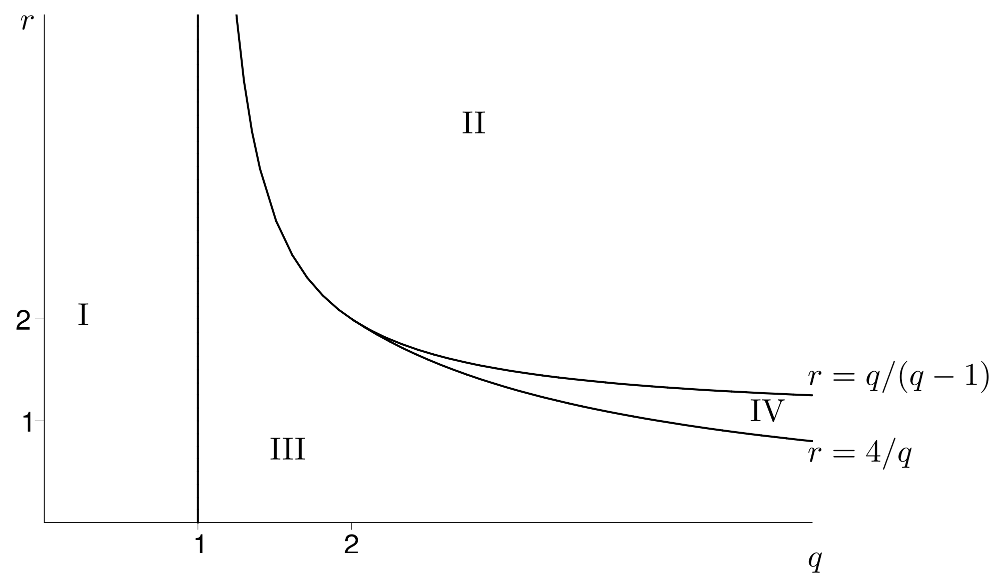
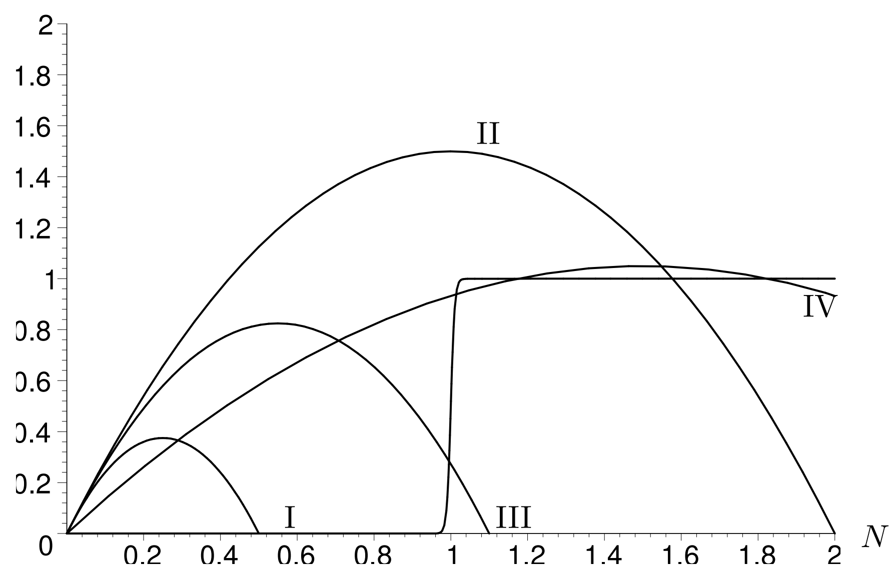


(c) (8 points) Above find a schematic of the q-r parameter plane which is divided into regions. In different regions, the number and/or (general) location of the steady states is different. Identify what is happening in each of the labeled regions, and verify that the curves listed are the appropriate boundaries of those regions.

Solution. To find the steady states, we must find the points where $\dot{N} = 0$. Using (A), we see that these points are given by

$$rN\left(1 - \frac{N}{q}\right) = \frac{1}{2}\left[1 + \tanh\left(\frac{N-1}{\epsilon}\right)\right]. \tag{B}$$



Left- and right-hand sides of (B) for $\epsilon = 0.01$.
The parabolas are labeled by region.

(Refer to the figure, which shows examples from each region.) For the case where $1-N$ is positive and $O(1)$, the right-hand side of (F) vanishes and we have

$$N\left(1-\frac{N}{q}\right)=0,$$

which means that the steady states are $N_*=0$, $N_*=q$. But this second state is less than 1 only when $q<1$. So when $q<1$ (region I), we have only one positive steady state, which is less than 1. Next we consider the case when $q>1$. For there to be a steady state near $N=1$, the parabola's value at 1 must be between 0 and 1, so

$$0<r\left(1-\frac{1}{q}\right)<1 \qquad \Longrightarrow \qquad r<\frac{q}{q-1}. \tag{C}$$

Therefore, if $r>q/(q-1)$ and $q>1$ (region II), there is no intersection near $N=1$. Thus there is only one positive steady state with $N_*>1$.

If (C) is satisfied, then there are either 1 or 3 positive steady states, depending on whether the parabola also intersects the upper part of the tanh. The maximum of the left-hand side occurs at $N_*=q/2$, and has the value $rq/4$. Therefore, for there to be three positive steady states, $rq>4$ (so the maximum is high enough) and $q>2$ (so that the upper branch of the tanh is intersected on both the up and down slopes). This is region IV. If either of these conditions are violated, we have only one positive steady state with $N_*\approx 1$ (region III).

In summary, $N=0$ is always a steady state and:
Region I: One positive steady state $N_*<1$.
Region II: One positive steady state $N_*>1$.
Region III: One positive steady state $N_*\approx 1$.
Region IV: Three positive steady states: two greater than 1 and one approximately 1.

(d) (3 points) Classify each of the steady states as stable or unstable.

Solution. We know that the stability of the steady state is determined by $f'(N_*)$, where N_* is the steady state and f is the right-hand side of (A). Therefore, from our graph we see that the roots alternate stability. In other words, zero is always unstable. The smallest positive steady state is stable. If there are three positive steady states, the largest positive steady states is stable, and the one between is unstable.

2. (3 points) When a population gets small, the members may find it difficult to locate mates, causing the growth rate to be negative. This phenomenon is called the *Allee effect.* Discuss how the model

$$\dot{N}=rN\left(1-\frac{N}{q}\right)(N-a), \qquad 0<a<q, \tag{2.1}$$

explains this effect, and what effects the additional term has on the number and stability of steady states.

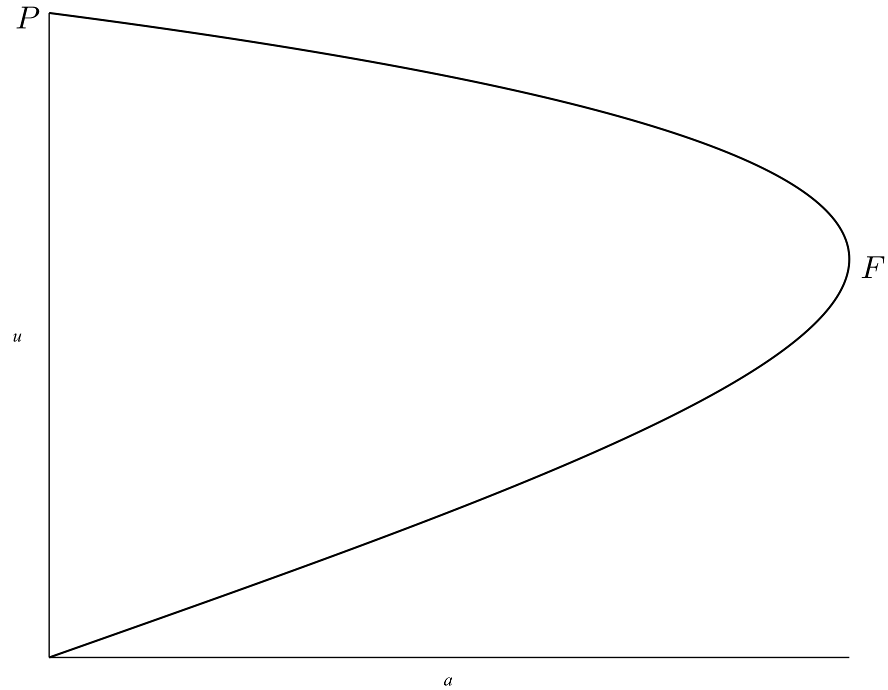
Solution. Let a be the threshold to find enough mates. Note that $N=a$ is now an additional steady state. If $N<a$, the last parenthetical term is negative while the other

terms are positive, so $\dot{N} < 0$. Hence the origin is now a stable steady state. If $N > a$, then all the terms are positive, and $\dot{N} > 0$. Therefore, $N = a$ is an unstable steady state.

3. (6 points) The equation

$$\frac{du}{dt} = u(\lambda - u) - ae^{-u}. \tag{2.2}$$

has the bifurcation diagram shown below. Determine the coordinates of the intersection point P and the fold point F. (For the fold point, you should find an explicit form for u_{F}, but you may write a_{F} in terms of u_{F}.)



Bifurcation diagram of (2.2).

Solution. The equilibrium solution must satisfy

$$f_{\mathrm{L}}(u) \equiv u(\lambda - u) = ae^{-u} \equiv f_{\mathrm{R}}(u). \tag{D}$$

When $a = 0$, the solutions are $u = 0$ and $u = \lambda$, so P must have the coordinates $(0, \lambda)$. At the fold point, there is exactly one solution. Hence the curves $f_{\mathrm{L}}(u)$ and $f_{\mathrm{R}}(u)$ cannot intersect transversely (because then there would be a second solution). Hence their derivatives must also match, as shown in the sketch below.

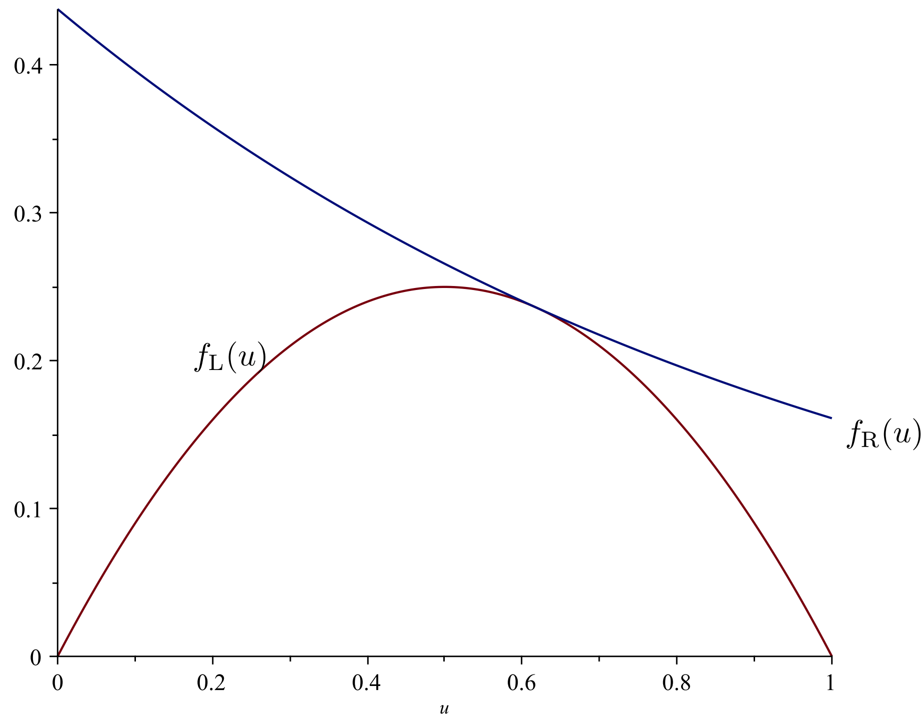
Therefore, for the fold case we must solve (D) together with its derivative:

$$\lambda - 2u = -ae^{-u} = -f_{\mathrm{R}}(u) = -f_{\mathrm{L}}(u) = -u(\lambda - u),$$

where we have used the fact that (D) holds at the equilibrium solution. Solving the above, we have

$$u^2 + (2 - \lambda)u - \lambda = 0 \tag{E}$$

$$u_{\mathrm{F}} = \frac{\lambda - 2 \pm \sqrt{(\lambda - 2)^2 + 4\lambda}}{2} = \frac{\lambda - 2 + \sqrt{\lambda^2 + 4}}{2}.$$



Sketch of $f_{\rm L}(u)$ and $f_{\rm R}(u)$ for $\lambda = 1$ (fold case).

(Note from the form of the quadratic equation that there will always be just one positive root.) Then solving (D) for a, we have

$$a_{\rm F} = e^{u_{\rm F}} u_{\rm F}(\lambda - u_{\rm F}). \tag{F}$$

As an alternative way of computing $u_{\rm F}$, we note that if we switch the dependence and think of a as a function of u, then $a'(u) = 0$ at the fold. Hence taking the derivative of (E) with respect to u, we have

$$\begin{aligned} a'(u) = e^{u}[u(\lambda - u) + (\lambda - 2u)] &= 0 \\ -u^2 + u(\lambda - 2) + \lambda &= 0, \end{aligned}$$

which is exactly the negative of (E).

4. Consider the following equation:

$$\frac{du}{dt} = (u - \lambda)(u^2 - \lambda) + \epsilon. \tag{2.3}$$

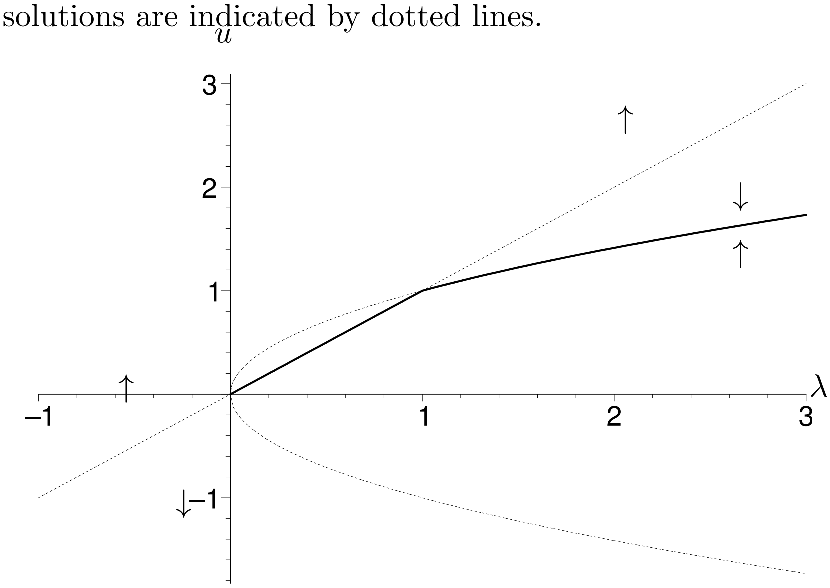
(a) (5 points) Draw the bifurcation diagram for $\epsilon = 0$. Classify and identify the location of each of the bifurcation points. Label each of the solution curves, indicate the stability of each branch, and indicate the direction of the flow.

Solution. With $\epsilon = 0$, clearly $u = \lambda$ is always a solution, and $u_\pm = \pm\sqrt{\lambda}$ is a solution for $\lambda \geq 0$. Therefore, there is clearly a pitchfork bifurcation at $\lambda = 0$, $u = 0$. We note that the two solutions $u = \lambda$ and $u = \sqrt{\lambda}$ intersect at $\lambda = 1$, $u = 1$, so there must be a transcritical bifurcation there. For $\lambda < 0$, we see that $\operatorname{sgn}(du/dt) = \operatorname{sgn}(u - \lambda)$, so $u = \lambda$ must be unstable. For $\lambda > 0$, we may rewrite (2.3) in this case as

$$\frac{du}{dt} = (u - \lambda)(u - u_+)(u - u_-).$$

Therefore, we have that du/dt is positive as $u \to \infty$, and negative as $u \to -\infty$. Therefore, the outer two solutions are unstable, while the inner solution is stable.

A bifurcation diagram is shown below. Stable solutions are indicated by solid lines, while unstable solutions are indicated by dotted lines.



Bifurcation diagram of (2.3) with $\epsilon = 0$.

(b) (8 points) Sketch the bifurcation diagrams for $\epsilon > 0$ and $\epsilon < 0$. Indicate the stability of each branch, and indicate the direction of the flow. Do **NOT** attempt to solve for the functional forms of the curves.

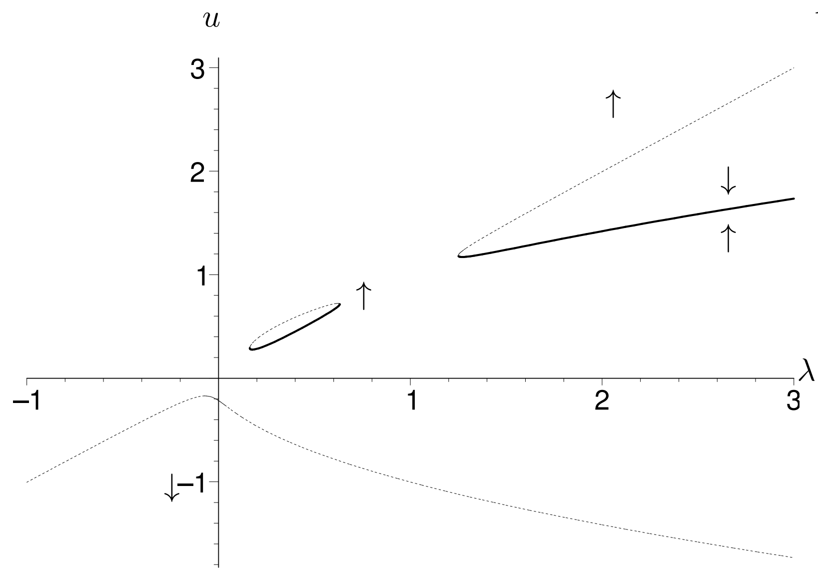
Solution. We note that at $u = 0$, equation (2.3) becomes

$$\frac{du}{dt} = \lambda^2 + \epsilon. \tag{G.1}$$

Therefore, we see that for $\epsilon > 0$, there are no λ-intercepts. Thus the upper two branches of the pitchfork must merge together into a fold, and the lower and left branch of the pitchfork must merge. Similarly, at $u = 1$, equation (2.3) becomes

$$\frac{du}{dt} = (1 - \lambda)^2 + \epsilon. \tag{G.2}$$

Therefore, we see that for $\epsilon > 0$, there are no curves which cross the line $u = 1$. Therefore, the left two branches near the transcritical bifurcation merge into a fold, as do the right



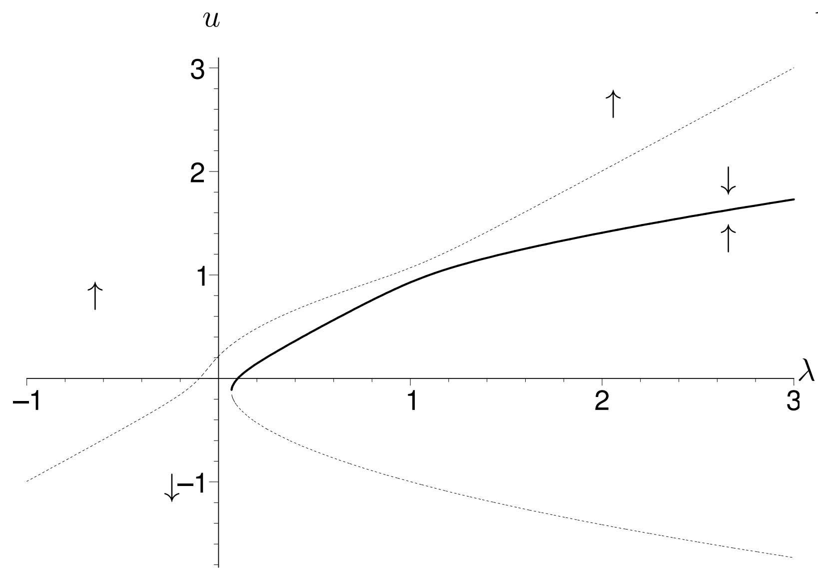
Bifurcation diagram of (2.3) with $\epsilon = 0.01$.

two branches. Since solution branches must connect, we see that the fold from the first pitchfork and the left fold from the second pitchfork must merge into a loop.

A bifurcation diagram is shown below. Stable solutions are indicated by solid lines, while unstable solutions are indicated by dotted lines.

For $\epsilon < 0$, the arguments are reversed. From (G.1), we have that there are two λ-intercepts, so the lower two branches of the pitchfork merge into a fold, and the upper and left branches merge. Similarly, from (G.2) we see that there are two curves which cross the line $u = 1$. Therefore, the top two branches near the transcritical bifurcation merge, as do the bottom two branches. Since solution branches must connect, we see that the fold from the first pitchfork and the bottom fold from the second pitchfork must merge into a single fold, and the other two trajectories must merge as well.

A bifurcation diagram is shown below. Stable solutions are indicated by solid lines, while unstable solutions are indicated by dotted lines.



Bifurcation diagram of (2.3) with $\epsilon = -0.01$.