MATH 616-010 Modeling in Applied Mathematics
Prof. D. A. Edwards Due: Sept. 13, 2023

# Homework Set 1 Solutions

1. The *Navier-Stokes* equations are the governing equations for much of fluid mechanics. In dimensional form, they are given by

$$\rho\left(\frac{\partial\tilde{\mathbf{u}}}{\partial\tilde{t}}+\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{\mathbf{u}}\right)=-\tilde{\nabla}\tilde{p}+\mu\tilde{\nabla}^2\tilde{\mathbf{u}},$$

where $\rho$ is the density, $\tilde{\mathbf{u}}$ is the velocity, $\tilde{p}$ is the pressure, and $\mu$ is the bulk viscosity. The $\tilde{\nabla}$ indicates that the spatial derivatives are taken with respect to variables with dimensions.

(a) (4 points) Write the units of each of the quantities listed in the Navier-Stokes equations. Check your work by verifying that each of the terms in the equation have the same units.

*Solution.*

$$[\rho]=ML^{-3},\qquad[\tilde{\mathbf{u}}]=LT^{-1},\qquad[\tilde{t}]=T,\qquad[\tilde{p}]=ML^{-1}T^{-2},\qquad[\mu]=ML^{-1}T^{-1}.$$

To check our work, we substitute these expressions and find that

$$\left[\rho\frac{\partial\tilde{\mathbf{u}}}{\partial\tilde{t}}\right]=\left[\rho\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{\mathbf{u}}\right]=\left[\tilde{\nabla}\tilde{p}\right]=\left[\mu\tilde{\nabla}^2\tilde{\mathbf{u}}\right]=\frac{M}{L^2T^2}.$$

(b) (6 points) Given a characteristic velocity $U$ and length scale $L$, scale the equations. On how many dimensionless parameters does your solution depend?

*Solution.* Divide the relevant length scales by $L$ and let

$$\tilde{\mathbf{u}}=U\mathbf{u},\qquad\tilde{t}=\frac{L}{U}t,\qquad\tilde{p}=\rho U^2p=\frac{\mu U}{L}q.$$

Note there are two possible choices for the nondimensionalization of $\tilde{p}$. However, since $U$ was given as our characteristic velocity, we should normalize $\tilde{\mathbf{u}}$ by it, rather than by other combinations of our parameters. Using these choices, we have

$$\frac{\rho U^2}{L}\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\frac{\rho U^2}{L}\nabla p+\frac{\mu U}{L^2}\nabla^2\mathbf{u}$$

$$\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\frac{\mu}{\rho UL}\nabla^2\mathbf{u}. \tag{A}$$

$$\frac{\rho UL}{\mu}\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla q+\nabla^2\mathbf{u}. \tag{B}$$

Our solution depends upon only one nondimensional parameter.

(c) (4 points) A characteristic *inertial force* $F_{\mathrm{i}}$ depends on the density of the fluid $\rho$, the characteristic velocity $U$, and the characteristic length $L$. Calculate $F_{\mathrm{i}}$. Since we are calculating a characteristic force (*i.e.*, one by which you would divide the dimensional quantities in the problem), you may set any arbitrary constants equal to 1.

*Solution.* Matching units on the quantities, we see that the only way to "build" a force is to let

$$F_{\mathrm{i}} = \rho U^2 L^2$$

so that

$$[F_{\mathrm{i}}] = \left[\rho U^2 L^2\right] = L^2 \left(\frac{M}{L^3}\right)\left(\frac{L^2}{T^2}\right) = \frac{ML}{T^2}.$$

(d) (4 points) A characteristic *viscous force* $F_{\mathrm{v}}$ depends on the bulk viscosity of the fluid $\mu$, the characteristic velocity $U$, and the characteristic length $L$. Calculate $F_{\mathrm{v}}$.

*Solution.* Matching units on the quantities, we see that the only way to "build" a force is to let

$$F_{\mathrm{v}} = \mu U L.$$

(e) (2 points) The *Reynolds number* Re of a system is given by the ratio of the inertial force to the viscous force. Rewrite your answer to (b) using the Re notation.

*Solution.* By our answers to (c) and (d), we have that

$$\mathrm{Re} = \frac{F_{\mathrm{i}}}{F_{\mathrm{v}}} = \frac{\rho U^2 L^2}{\mu U L} = \frac{\rho U L}{\mu}.$$

Therefore, equations (A) and (B) become

$$\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}.$$

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla q + \nabla^2\mathbf{u}.$$

For any particular problem, we expect the dimensionless pressure to be $O(1)$. Therefore, we would use the $p$ equation when inertial forces dominate (and hence Re is large) and we would use the $q$ equation when viscous forces dominate (and hence Re is small).

2. (4 points) A spherical gas bubble with ratio of specific heats $\gamma$ (dimensionless) is surrounded by an infinite sea of liquid of density $\rho_l$. The bubble oscillates with growth and contraction periodically with small amplitude at a well-defined frequency $\omega$. Assuming a physical law

$$f(P, R, \rho_l, \omega, \gamma) = 0,$$

where $P$ is the mean pressure inside the bubble and $R$ is the mean radius, show that $\omega$ must vary inversely with $R$.

*Solution.* All of the relevant units have been listed above, except for

$$[\omega] = T^{-1}.$$

Other than $\gamma$, the only nondimensional quantity $\pi$ we can create is

$$\pi = \frac{P}{\rho\omega^2 R^2}, \qquad [\pi] = \left[\frac{P}{\rho\omega^2 R^2}\right] = \left(\frac{M}{LT^2}\right)\left(\frac{L^3}{M}\right)\left(\frac{T^2}{L^2}\right) = 1.$$

Therefore, we have that $\pi$ must be some dimensionless function of $\gamma$, and hence we have

$$\omega^2 = \frac{P}{\pi(\gamma)\rho R^2}$$
$$\omega = \frac{g(\gamma)}{R}\sqrt{\frac{P}{\rho}},$$

for some function $f$.

3. A ball of mass $m$ is tossed upward with initial velocity $V$. If $\tilde{v}$ is the velocity of the ball, we assume that the force casued by air resistance is proportional to $\tilde{v}|\tilde{v}|$ and that the gravitational field is constant.

   (a) (2 points) Formulate an initial value problem for the height $\tilde{x}$ of the ball at any time $\tilde{t}$.

*Solution.* We denote the height of the ball at time $\tilde{t}$ as $\tilde{z}(\tilde{t})$, and we denote the constant of proportionality in the air resistance law as $k$. Gravity always acts downward, so it is a negative acceleration. The air resistance term always opposes the direction of motion, so we have

$$m\frac{d^2\tilde{z}}{d\tilde{t}^2} = -mg - k\frac{d\tilde{z}}{d\tilde{t}}\left|\frac{d\tilde{z}}{d\tilde{t}}\right|. \tag{C.1}$$

In order for the units to balance, $[k] = M/L$. In addition, we have the initial data, which is given by

$$\tilde{z}(0) = 0, \qquad \frac{d\tilde{z}}{d\tilde{t}}(0) = V. \tag{C.2}$$

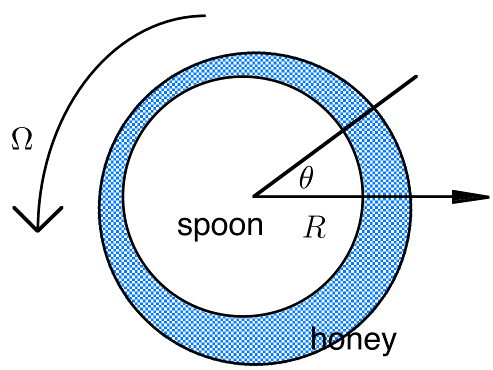
   (b) (4 points) Using your notes from class, choose meaningful characteristic length and time scales and recast the problem in dimensionless form. Your initial conditions should be free of all parameters.

*Solution.* We would like to choose our length and time scales such that the scaled velocity condition is 1. We also expect gravity to be the dominant force, so as in class we choose $V^2/g$ to be the characteristic length scale and $V/g$ to be the characteristic time scale. Therefore, we let

$$\tilde{z}(\tilde{t}) = \frac{V^2 z(t)}{g}, \qquad \tilde{t} = \frac{Vt}{g}.$$

Making these substitutions into (C), we obtain

$$
\begin{aligned}
m\frac{V^2/g}{V^2/g^2}\frac{d^2z}{dt^2} &= -mg - k\left(\frac{V^2/g}{V/g}\right)^2 \frac{dz}{dt}\left|\frac{dz}{dt}\right| \\
mg\frac{d^2z}{dt^2} &= -mg - kV^2\frac{dz}{dt}\left|\frac{dz}{dt}\right| \\
\frac{d^2z}{dt^2} &= -1 - \beta\frac{dz}{dt}\left|\frac{dz}{dt}\right|, \qquad \beta = \frac{kV^2}{mg}, \\
z(0) &= 0, \\
\frac{V^2/g}{V/g}\frac{dz}{dt}(0) &= V \\
\frac{dz}{dt}(0) &= 1.
\end{aligned}
$$




4. Suppose that a person dips the end of a wooden spoon of radius $R$ into a viscous fluid (honey, for instance), then removes the spoon and holds it horizontal. The person then twirls the end of the spoon around with angular velocity $\Omega$ (see figure). The equation for the velocity $\tilde{u}$ of the fluid in the $\theta$-direction is given by

$$\nu\frac{\partial^2\tilde{u}}{\partial\tilde{z}^2} = g\cos\theta, \tag{1.1}$$

where $\nu$ is kinematic viscosity and $\tilde{z}$ is distance measured in the direction normal to the surface of the spoon. Here $\tilde{z} = 0$ corresponds to the surface of the spoon.

(a) (2 points) What is the velocity of the honey at the surface of the spoon?

*Solution.* The velocity of the honey at the surface of the spoon must be the same as the velocity at the surface of the spoon itself, which is $\Omega R$.

(b) (2 points) If $H$ is some characteristic height of the honey, find the characteristic acceleration due to the rotation of the spoon.

*Solution.* Since the acceleration due to gravity is listed on the right, the acceleration due to the rotation must be the quantity on the left-hand side of (1.1). If we use the characteristic velocity from (a), we have

$$\tilde{u}(\tilde{z}) = \Omega R u(z), \qquad \tilde{z} = Hz.$$

Making these substitutions into (1.1), we obtain

$$\frac{\nu\Omega R}{H^2}\frac{\partial^2 u}{\partial z^2} = g\cos\theta, \tag{D}$$

and hence a characteristic acceleration due to rotation must be $\nu\Omega R/H^2$.

(c) (2 points) Find the dimensionless parameter on which the solution must depend.

*Solution.* Continuing to simplify (D), we obtain

$$\alpha\frac{\partial^2 u}{\partial z^2} = \cos\theta, \qquad \alpha = \frac{\nu\Omega R}{gH^2}. \tag{E}$$

Since (E) is dimensionless, the solution must depend only on $\alpha$.

(d) (4 points) How does the solution behave as $\Omega \to \infty$? How does it behave as $\Omega \to 0$? Classify each of the steady states as stable or unstable.

*Solution.* As $\Omega \to \infty$, $\alpha \to \infty$. Therefore, we see that if we rewrite (E) as

$$\frac{\partial^2 u}{\partial z^2} = \alpha^{-1}\cos\theta,$$

we see that $u$ must become nearly linear. This state is stable. As $\Omega \to 0$, $\alpha \to 0$. Therefore, (E) nearly becomes $\cos\theta = 0$, which would mean there would be fluid only near $\theta = \pm\pi/2$, which corresponds to the physical case of fluid dripping off the spoon. This state is unstable.