Updates

1. Topics are due Monday, Sept. 25.

Homework Set 3 (Revised)

Read sections 2.1–2.4.

The Phase Plane

1. Consider the following ordinary differential equation:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x - \sqrt{|x|} = 0. \tag{3.1}$$

   (a) (2 points) Rewrite (3.1) as a set of two coupled first-order ODEs.
   (b) (2 points) Find the equilibrium points.
   (c) (2 points) Describe the behavior of solutions near the equilibrium points.
   (d) (3 points) Using some sort of computer software, sketch the phase plane. Verify that your results agree with your analysis in parts (b) and (c).

2. Consider the following phase-plane system:

$$\dot{x} = -\delta x - \mu y + xy, \qquad \delta \neq 0, \tag{3.2a}$$

$$\dot{y} = \mu x - \delta y + \frac{x^2 - y^2}{2}, \quad \mu \neq 0. \tag{3.2b}$$

   (a) (4 points) Show that there are four equilibrium points. You should find that one is the origin. Do **NOT** solve explicitly for the other three; just write the equation $f(x) = 0$ that they satisfy and verify that real solutions exist. (*Hint: Calculate $f(x_*)$, where x_* is an extremum of f.*)
   (b) (3 points) Classify the fixed point at the origin and show that its stability depends only on δ.
   (c) (4 points) Let r_i ($i = 1, 2, 3$) be the distance of the ith other fixed point from the one at the origin. Show that $r_i \geq 2|\delta|$ and $r_i \geq 2|\mu|$. (*Hint: It may be helpful to write all trig functions in terms of 3θ.*)
   (d) (3 points) Show that the other three fixed points are all saddle points.

Polar Coordinates

3. Consider the following system in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \cos 4\theta.$$

(a) (2 points) Find all the curves where *either* $\dot{r} = 0$ or $\dot{\theta} = 0$.
(b) (3 points) Find and examine the stability of all nine equilibrium points. (*Hint: Use $\dot{r}$ and $\dot{\theta}$; don't linearize in (x, y).*)
(c) (3 points) Using some sort of computer software, sketch the phase plane in Cartesian coordinates. Verify that your results agree with your analysis in part (b).