Laplace Transforms (4/12 Version)

The *Laplace transform* of a function $f(t)$ is given by

$$\mathcal{L}\{f(t)\} = \hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

If we are taking the Laplace transform of a function of multiple variables, it gets rid of the t-part:

$$\mathcal{L}\{f(x,t)\} = \hat{f}(x),$$

and the s is treated as a parameter.

It is useful for simplifying derivatives in time:

$$\mathcal{L}\{\dot{f}\} = s\hat{f}(s) - f(0), \tag{1a}$$

$$\mathcal{L}\{\ddot{f}\} = s^2\hat{f}(s) - sf(0) - \dot{f}(0), \tag{1b}$$

where the dot denotes a time derivative. We also computed some useful properties, the *shifting properties*:

$$\mathcal{L}\{e^{at}f(t)\} = \hat{f}(s-a), \tag{3a}$$

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}\hat{f}(s), \tag{3b}$$

and the *convolution property*:

$$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\, d\tau\right\} = \hat{f}(s)\hat{g}(s). \tag{4}$$

We also computed the following Laplace transform pairs:

$$e^{-t} \iff \frac{1}{s+1}, \tag{2}$$

$$\frac{\lambda}{2t^{3/2}} \exp\left(-\frac{\lambda^2}{4t}\right) \iff e^{-\lambda\sqrt{s}}\sqrt{\pi}, \quad \lambda > 0, \tag{5}$$

$$1 \iff \frac{1}{s}, \tag{6}$$

$$\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) \iff \frac{e^{-x\sqrt{s}}}{s}, \quad x \geq 0. \tag{7}$$