MATH 351-010 Engineering Mathematics I
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Facts About Vectors

A *vector space* is an infinite collection of objects (vectors) that satisfy the ten properties given in class. (The most common is $\mathcal{R}^n$.) A *subspace* S is an infinite sub*set* of the vector space that satisfies these properties:

$$\mathbf{v}_1, \mathbf{v}_2 \in S \implies \mathbf{v}_1 + \mathbf{v}_2 \in S \tag{1}$$

$$\mathbf{v} \in S \implies c\mathbf{v} \in S, \quad c \in \mathcal{R}, \tag{2}$$

$$\mathbf{0} \in S. \tag{8}$$

If asked to prove that S is a subspace, use the special property/characteristic that defines S to turn the statement on the right-hand side of the arrow into an actual equation you can work with. Then use the property again with the statements on the left-hand side of the arrow to simplify.

Recall that

$$\operatorname{Span} B = \{\textit{all} \text{ linear combinations of vectors } \mathbf{b}_j \in B\} = \left\{ \sum_{j=1}^{n} c_j \mathbf{b}_j,\ c \in \mathcal{R} \right\}.$$

Note that B has n vectors, while the span is infinite. If $\dim V = n$, then B is a *basis* for V if it satisfies *two* of the following three properties:

1. B has n vectors,
2. $\operatorname{Span} B = V$,
3. B is linearly independent.

Typically one of the first two can be inferred from the problem, which means that you have to verify that B is *linearly independent*.

There are two tests for this. First, there's the computational test, where you check if one of the vectors is a linear combination of the others:

$$\text{Does } \mathbf{b}_n = \sum_{j=1}^{n-1} c_j \mathbf{b}_j \text{ have a solution?} \qquad \begin{cases} \text{Yes}: & \text{linearly dependent} \\ \text{No}: & \text{linearly independent} \end{cases}$$

This is usually best if you have actual numbers to try, and a small number of vectors. Otherwise, use the theoretical test:

$$\text{What is the solution of } \sum_{j=1}^{n} c_j \mathbf{b}_j = \mathbf{0}? \qquad \begin{cases} \text{Some } c_j \neq 0: & \text{linearly dependent} \\ \text{Just } c_j = 0: & \text{linearly independent} \end{cases}$$

(If the vectors are linearly dependent, this test will give you free variables.)

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