Homework Set 9 Solutions

1. Consider the following matrix and vectors:

$$C_1 = \begin{pmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{pmatrix}, \qquad \mathbf{z}_1 = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}, \quad \mathbf{z}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{z}_3 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

By direct multiplication, show that the listed vectors $\mathbf{z}_i$ are eigenvectors for C_1, and find the corresponding eigenvalues.

Solution. Doing the multiplications, we obtain

$$C_1\mathbf{z}_1 = \begin{pmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} = \mathbf{z}_1,$$

$$C_1\mathbf{z}_2 = \begin{pmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = -\mathbf{z}_2,$$

$$C_1\mathbf{z}_3 = \begin{pmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} = 2\mathbf{z}_3.$$

Therefore, we have that $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$.

2. Let λ_j be the jth eigenvalue of A, $\mathbf{z}_j$ be the eigenvector of A corresponding to λ_j. In addition, let α be a real number and β_j be the jth eigenvalue of B. In each part below, show that $\mathbf{z}_j$ is also an eigenvector for B corresponding to β_j and prove the listed identity.

(a) If $B = \alpha A$, show that $\beta_j = \alpha\lambda_j$.

Solution.

$$B\mathbf{z}_j = \alpha A\mathbf{z}_j = \alpha\lambda_j\mathbf{z}_j = \beta_j\mathbf{z_j}.$$

(b) If $B = A + \alpha I$, show that $\beta_j = \lambda_j + \alpha$.

Solution.

$$B\mathbf{z}_j = (A + \alpha I)\mathbf{z}_j = A\mathbf{z}_j + \alpha\mathbf{z}_j = \lambda_j\mathbf{z}_j + \alpha\mathbf{z}_j = \beta_j\mathbf{z_j}.$$

(c) If $B = p(A)$, where $p(x)$ is a polynomial in x, show that $\beta_j = p(\lambda_j)$.

Solution. From notes in class, we have that $A^m\mathbf{z}_j = \lambda^m\mathbf{z}_j$. Therefore, letting

$$p(A) = \sum_{m=0}^{n} c_m A^m,$$

we have

$$B\mathbf{z}_j = p(A)\mathbf{z}_j = \left(\sum_{m=0}^{n} c_m A^m\right)\mathbf{z}_j = \sum_{m=0}^{n} c_m \lambda^m \mathbf{z}_j = p(\lambda_j)\mathbf{z}_j = \beta_j \mathbf{z}_j.$$

3. Consider the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of A.

Solution. Since A is triangular, we have from notes in class that the eigenvalues are the diagonal entries, so $\lambda_1 = 3$, $\lambda_2 = 2$.

(b) Find the eigenvectors of A.

Solution. Solving $(A - \lambda_1 I)\mathbf{z}_1 = \mathbf{0}$ in augmented matrix form, we have

$$\begin{bmatrix} 3-3 & -1 & 0 \\ 0 & 2-3 & \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad y = 0, \qquad x \text{ free.}$$

Therefore, choosing $x = 1$ for simplicity, we see that a typical eigenvector is $(1, 0)^T$. Solving $(A - \lambda_2 I)\mathbf{z}_2 = \mathbf{0}$ in augmented matrix form, we have

$$\begin{bmatrix} 3-2 & -1 & 0 \\ 0 & 2-2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Longrightarrow x - y = 0, \qquad y \text{ free.}$$

Therefore, choosing $y = 1$ for simplicity, we see that a typical eigenvector is $(1, 1)^T$.

4. Repeat the steps of #3 for the matrix

$$A^T = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}.$$

Discuss any similarities and differences.

Solution. Since A^T is also triangular, we have that the eigenvalues are the same diagonal entries as in A, so the eigenvalues are the same. Solving $(A^T - \lambda_1 I)\mathbf{z}_1 = \mathbf{0}$ in augmented matrix form, we have

$$\begin{bmatrix} 3-3 & 0 & 0 \\ -1 & 2-3 & \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad x + y = 0, \qquad y \text{ free.}$$

Therefore, choosing $y = 1$ for simplicity, we see that a typical eigenvector is $(-1, 1)^T$. Solving $(A^T - \lambda_2 I)\mathbf{z}_2 = \mathbf{0}$ in augmented matrix form, we have

$$\begin{bmatrix} 3-2 & 0 & 0 \\ -1 & 2-2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad x = 0, \qquad y \text{ free.}$$

Therefore, choosing $y = 1$ for simplicity, we see that a typical eigenvector is $(0, 1)^T$. Hence the eigenvectors of A and A^T are different.

5. Let $A \in \mathcal{R}^{n\times n}$, $\lambda \neq 0$ an eigenvalue for A. Show that if $\mathbf{x}$ an eigenvector for λ, $\mathbf{x} \in \operatorname{col} A$.

Solution. Using the definition of matrix-vector multiplication, we have

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ \sum_{j=1}^{n} x_j \mathbf{a}_j &= \lambda\mathbf{x} \\ \mathbf{x} &= \frac{1}{\lambda}\sum_{j=1}^{n} x_j\mathbf{a}_j, \end{aligned}$$

where $\mathbf{a}_j$ is the jth column of A. So $\mathbf{x}$ is a linear combination of the columns, and hence $\mathbf{x} \in \operatorname{col} A$.

6. Let A be a matrix whose columns all add up to a fixed constant δ. Show that δ is an eigenvalue of A.

Solution. Consider the matrix $A - \delta I$. Since we are subtracting δ from each column, we see that the sum of each column of $A - \delta I$ is zero. Thus the sum of the rows of $A - \delta I$ is a row of zeroes. Since a linear combination of the rows forms the zero vector, the rows of $A - \delta I$ are linearly dependent, and $A - \delta I$ is not invertible. Thus $\det(A - \delta I) = 0$, and δ is an eigenvalue for A.

7. Find the eigenvalues and corresponding eigenspaces for the following matrices:

$$A_1 = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}. \tag{9.1}$$

Solution. Calculating the characteristic polynomial of A_1, we have

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 5 &= 0 \qquad \text{(A)} \\ \lambda_\pm &= \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i. \end{aligned}$$

Solving for the eigenspaces, we have

$$[A_1 - (2+i)I]\mathbf{z}_1 = \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}\mathbf{z}_1 = \mathbf{0}.$$

The first row is $-(1+i)$ times the first, so we see that $x + (1-i)y = 0$, or

$$\operatorname{Span}\mathbf{z}_+ = \begin{pmatrix} -(1-i)y \\ y \end{pmatrix}.$$

We know that the eigenvectors are complex conjugates of one another, so

$$\operatorname{Span}\mathbf{z}_- = \begin{pmatrix} -(1+i)y \\ y \end{pmatrix}.$$

Calculating the characteristic polynomial of A_2, we have

$$\begin{vmatrix} (7-\lambda) & 4 \\ -3 & (-1-\lambda) \end{vmatrix} = -7 - 6\lambda + \lambda^2 + 12 = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5) = 0.$$

Therefore, we have that $\lambda_1 = 1$, $\lambda_2 = 5$. Solving for the eigenspaces, we have

$$(A_2 - I)\mathbf{z}_1 = \begin{pmatrix} 6 & 4 \\ -3 & -2 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

The rows are multiples of one another, so we see that $-3x - 2y = 0$, or

$$\operatorname{Span}\mathbf{z}_1 = \begin{pmatrix} x \\ -3x/2 \end{pmatrix}.$$

Similarly, we have

$$(A_2 - 5I)\mathbf{z}_2 = \begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

The rows are multiples of one another, so we see that $2x + 4y = 0$, or

$$\operatorname{Span}\mathbf{z}_2 = \begin{pmatrix} -2y \\ y \end{pmatrix}.$$

8. Consider the following matrix:

$$C = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(Matrices of this form occur regularly in physical problems. For instance, this matrix could represent the diffusion of heat in a one-dimensional bar.) Find the eigenvalues and a basis for each of the corresponding eigenspaces.

Solution. Calculating the characteristic polynomial, we have

$$\det(C - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ -1 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)[(2-\lambda)(1-\lambda) - 1] + \lambda - 1 = (1-\lambda)(1 - 3\lambda + \lambda^2) + \lambda - 1$$
$$= -3\lambda + 4\lambda^2 - \lambda^3 = -\lambda(\lambda-3)(\lambda-1) = 0.$$

Therefore, we have that $\lambda_1 = 1$, $\lambda_2 = 0$, and $\lambda_3 = 3$. Solving for the eigenspaces, we obtain

$$(C - \lambda_1 I)\mathbf{z}_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

The first and third rows are the same, so we see that $y = 0$, which implies that $x = -z$, and we have

$$\operatorname{Span} \mathbf{z}_1 = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix}.$$

$$(C - \lambda_2 I)\mathbf{z}_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

Row reducing, we have

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} \\ \text{b} + \text{a} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

The second and third rows are multiples of one another, so we see that $y - z = 0$, which implies that $x = z$, and we have

$$\operatorname{Span} \mathbf{z}_2 = \begin{pmatrix} z \\ z \\ z \end{pmatrix}.$$

Lastly, we have

$$(C - \lambda_3 I)\mathbf{z}_3 = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{z}_3 = \mathbf{0}.$$

Row reducing, we obtain

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} - \text{b} \\ \text{a} - 2\text{b} \\ \text{c} \end{matrix} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{pmatrix}$$

The second and third rows are multiples of one another, so we see that $y + 2z = 0$ and $-x + z = 0$, so the result is

$$\operatorname{Span} \mathbf{z}_3 = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix}.$$

9. Find the eigenvalues and eigenvectors of the matrix

$$D = \begin{pmatrix} \alpha & -1 \\ \alpha - 1 & 0 \end{pmatrix}.$$

Solution. Calculating the characteristic polynomial of D, we have

$$\begin{aligned} p_D(\lambda) &= \begin{vmatrix} (\alpha - \lambda) & -1 \\ \alpha - 1 & -\lambda \end{vmatrix} = -\alpha\lambda + \lambda^2 + (\alpha - 1) = \lambda^2 - \alpha\lambda + (\alpha - 1) \\ &= [\lambda - (\alpha - 1)](\lambda - 1). \end{aligned}$$

Therefore, we have that $\lambda_1 = 1$, $\lambda_2 = \alpha - 1$. Solving for the eigenspaces, we have

$$(D + I)\mathbf{z}_1 = \begin{pmatrix} \alpha - 1 & -1 \\ \alpha - 1 & -1 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

The rows are identical, so we see that $(\alpha - 1)x - y = 0$, so a typical eigenvector is

$$\mathbf{z}_1 = \begin{pmatrix} 1 \\ \alpha - 1 \end{pmatrix}.$$

Similarly, we have

$$(D - (\alpha - 1)I)\mathbf{z}_2 = \begin{pmatrix} 1 & -1 \\ \alpha - 1 & -(\alpha - 1) \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

The rows are multiples of one another, so we see that $x - y = 0$, so a typical eigenvector is

$$\mathbf{z}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

10. The *Cayley-Hamilton theorem* states that every matrix $A \in \mathcal{R}^{n \times n}$ is a root of its characteristic polynomial. So if

$$p_A(\lambda) = \sum_{j=0}^{n} a_j \lambda^j,$$

then

$$p_A(A) = \sum_{j=0}^{n} a_j A^j = O.$$

(Note that $A^0 = I$.) Verify that the matrix A_1 in (9.1) satisfies its characteristic polynomial.

Solution. We have already solved for the characteristic polynomial [see (A)]. Thus we wish to calculate

$$\begin{aligned} p_{A_1}(A_1) &= A_1^2 - 4A_1 + 5I \\ &= \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} - 4 \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -8 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 8 \\ -4 & -7 \end{pmatrix} = O, \end{aligned}$$

as required.