Homework Set 8 Solutions

1. Show that the set P of all polynomials, with the normal definitions of addition and scalar multiplication, forms a vector space.

Solution. Let

$$p_i(x) = \sum_{j=0}^{n_i} a_{ij} x^j, \quad i \in \mathcal{Z}^+$$

be an element of P. Then

$$p_1(x) + p_2(x) = \sum_{j=0}^{n_1} a_{1j} x^j + \sum_{j=0}^{n_2} a_{2j} x^j = \sum_{j=0}^{n_2} a_{2j} x^j + \sum_{j=0}^{n_1} a_{1j} x^j = p_2(x) + p_1(x). \tag{A.3}$$

We note immediately that $p_1(x) + p_2(x) \in P$, so P is closed under addition (property 1).

$$\begin{aligned}[p_1(x) + p_2(x)] + p_3(x) &= \left(\sum_{j=0}^{n_1} a_{1j} x^j + \sum_{j=0}^{n_2} a_{2j} x^j \right) + \sum_{j=0}^{n_3} a_{3j} x^j \\ &= \sum_{j=0}^{n_1} a_{1j} x^j + \left(\sum_{j=0}^{n_2} a_{2j} x^j + \sum_{j=0}^{n_3} a_{3j} x^j \right) = p_1(x) + [p_2(x) + p_3(x)],\end{aligned} \tag{A.4}$$

$$\begin{aligned}\alpha[p_1(x) + p_2(x)] &= \alpha \left(\sum_{j=0}^{n_1} a_{1j} x^j + \sum_{j=0}^{n_2} a_{2j} x^j \right) = \alpha \sum_{j=0}^{n_1} a_{1j} x^j + \alpha \sum_{j=0}^{n_2} a_{2j} x^j \\ &= \alpha p_1(x) + \alpha p_2(x).\end{aligned} \tag{A.6}$$

We note immediately that $\alpha p_1(x) \in P$, so P is closed under scalar multiplication (property 2).

$$\begin{aligned}(\alpha + \beta) p_1(x) &= (\alpha + \beta) \sum_{j=0}^{n_1} a_{1j} x^j = \alpha \sum_{j=0}^{n_1} a_{1j} x^j + \beta \sum_{j=0}^{n_1} a_{1j} x^j \\ &= \alpha p_1(x) + \beta p_1(x),\end{aligned} \tag{A.7}$$

$$(\alpha\beta) p_1(x) = (\alpha\beta) \sum_{j=0}^{n_1} a_{1j} x^j = \alpha \left(\beta \sum_{j=0}^{n_1} a_{1j} x^j \right) = \alpha[\beta p_1(x)], \tag{A.5}$$

$$1 \cdot p_1(x) = \sum_{j=0}^{n_1} 1 \cdot a_{1j} x^j = \sum_{j=0}^{n_1} a_{1j} x^j = p_1(x). \tag{A.10}$$

Next we define the zero element to be the zero polynomial

$$p_0(x) = \sum_{j=0}^{\infty} 0x^j,$$

so we have

$$p_1(x) + p_0(x) = \sum_{j=0}^{n_1} a_{1j}x^j + \sum_{j=0}^{\infty} 0x^j = \sum_{j=0}^{n_1} a_{1j}x^j = p_1(x). \tag{A.8}$$

Letting

$$p_{-1}(x) = \sum_{j=0}^{n_1} -a_{1j}x^j,$$

we obtain

$$p_1(x) + p_{-1}(x) = \sum_{j=0}^{n_1} a_{1j}x^j + \sum_{j=0}^{n_1} -a_{1j}x^j = \sum_{j=0}^{\infty} 0x^j = p_0(x), \tag{A.9}$$

so $p_{-1}(x)$ is the additive inverse of $p_1(x)$. Hence we have satisfied all the properties.

2. Let A be a particular vector (matrix) in $\mathcal{R}^{2\times 2}$. Determine whether or not the following are subspaces of $\mathcal{R}^{2\times 2}$:

(a) $S_1 = \{B \in \mathcal{R}^{2\times 2} | AB = BA\}$.

Solution. We check the properties one at a time. First, we check property 2: if $B \in S_1$, is $\alpha B \in S_1$?

$$A(\alpha B) = \alpha AB = \alpha(BA) = (\alpha B)A,$$

where in the third step we have used the fact that $B \in S_1$. So property 2 is satisfied. Now let $C \in S_1$, and let's check property (1). Note that

$$A(B + C) = AB + AC = BA + CA = (B + C)A,$$

where in the third step we have used the fact that B and C are in S_1. Hence $B + C \in S_1$ and hence property 1 is satisfied. Therefore S_1 is a subspace of $\mathcal{R}^{2\times 2}$. Lastly, we check property 8, where the zero vector is $O \in \mathcal{R}^{2\times 2}$. But

$$AO = O = OA, \tag{B}$$

so property 8 is satisfied.

(b) $S_2 = \{B \in \mathcal{R}^{2\times 2} | AB \neq BA\}$.

Solution. First we check property 8. But by (B), $AO = OA$, so $O \neq S_2$, so S_2 is not a subspace of $\mathcal{R}^{2\times 2}$.

(c) $S_3 = \{B \in \mathcal{R}^{2\times 2} | BA = O\}$.

Solution. First, we check property 2: if $B \in S_1$, is $\alpha B \in S_1$? Since

$$(\alpha B)A = \alpha(BA) = \alpha O = O,$$

we see that $B \in S_3$ implies $\alpha B \in S_3$, so property 2 is satisfied. Now let $C \in S_3$. Since

$$(B + C)A = BA + CA = O + O = O,$$

we see that $B + C \in S_3$, and hence property 1 is satisfied. Property 8 is satisfied because of (B), and so S_3 is a subspace of $\mathcal{R}^{2\times 2}$.

3. Let

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -2 & 4 \\ 3 & -7 & 10 \end{pmatrix}.$$

Calculate $\mathcal{N}(A)$ and $\mathcal{N}(A^T)$.

Solution. To calculate $\mathcal{N}(A)$, we solve the augmented matrix system $A\mathbf{x} = \mathbf{0}$:

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & 3 & -2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -7 & 10 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} \\ \text{d} = \text{b} - 2\text{a} \\ \text{e} = \text{b} - 3\text{c} \end{matrix} \begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & -8 & 8 & 0 \\ 0 & -16 & 16 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} + 3\text{d}/8 \\ -\text{d}/8 \\ 2\text{d} - \text{e} \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ x_3 &\text{ free.} \end{aligned} \quad \Longrightarrow \quad \mathcal{N}(A) = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix}.$$

To calculate $\mathcal{N}(A^T)$, we solve the augmented matrix system $A^T\mathbf{x} = \mathbf{0}$:

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 3 & -2 & -7 & 0 \\ -2 & 4 & 10 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} \\ \text{d} = 3\text{a} - \text{b} \\ \text{e} = \text{a} + 2\text{c} \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 8 & 16 & 0 \\ 0 & 8 & 16 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} - \text{d}/4 \\ \text{d}/8 \\ \text{d} - \text{e} \end{matrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_3 &\text{ free.} \end{aligned} \quad \Longrightarrow \quad \mathcal{N}(A^T) = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix}.$$

4. Let $C[0, 1]$ be the space of continuous functions defined for $0 \le x \le 1$. For each of the following, consider the subspace V spanned by the set of functions. Find a basis for V and the dimension of V.

(a) x, $x^2 + 1$, $x^2 - 1$

Solution. To check and see if the three vectors are linearly dependent, we have

$$c_1x + c_2x^2 + c_2 + c_3(x^2 - 1) = (c_2 + c_3)x^2 + c_1x + (c_2 - c_3) = 0.$$

The x term tells us that $c_1 = 0$. Solving the coefficients of the other two terms together, we have

$$\begin{aligned} c_2 + c_3 &= 0 \\ c_2 - c_3 &= 0 \end{aligned} \qquad \Longrightarrow \qquad c_2 = c_3 = 0,$$

and hence the three vectors are linearly independent. Therefore $\dim V = 3$ and the set $\{x, x^2+1, x^2-1\}$ provides a basis for V.

(b) $\sin x \cos x$, $\sin 2x$

Solution. To check and see if the two vectors are linearly dependent, we have

$$c_1 \sin x \cos x + c_2 \sin 2x = c_1 \sin x \cos x + c_2(2 \sin x \cos x) = 0,$$

where we have used a trigonometric identity. Hence the solution is $c_1 = -2c_2$, and the vectors are linearly dependent. Therefore, $\dim V = 1$ and either of $\sin x \cos x$ or $\sin 2x$ provides a basis for it.

(c) x, $\sin x$, $\cos x$

Solution. To check and see if the three vectors are linearly dependent, we have

$$c_1 x + c_2 \sin x + c_3 \cos x = 0.$$

But there is no way to write $\sin x$ and $\cos x$ in terms of x, so the only solution is $(c_1, c_2, c_3) = (0, 0, 0)$, and hence the three vectors are linearly independent. Therefore $\dim V = 3$ and the set $\{x, \sin x, \cos x\}$ provides a basis for V.

(d) e^x, e^{-x}, $\sinh x$

Solution. To check and see if the three vectors are linearly dependent, we have

$$\begin{aligned} c_1 e^x + c_2 e^{-x} + c_3 \sinh x = c_1 e^x + c_2 e^{-x} + c_3 \frac{e^x - e^{-x}}{2} &= 0 \\ \frac{(2c_1 + c_3)e^x}{2} + \frac{(2c_2 - c_3)e^{-x}}{2} &= 0, \end{aligned}$$

Solving the coefficients together, we have

$$\begin{aligned} 2c_1 + c_3 &= 0 \\ 2c_2 - c_3 &= 0, \end{aligned}$$

which is two equations in three unknowns. Hence for any c_3 you can find a solution, so the three vectors are linearly independent. Therefore, $\dim V = 2$ and any two of the three vectors $\{e^x, e^{-x}, \sinh x\}$ form a basis for V.

5. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 9 \\ -1 \\ -2 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) What is the dimension of $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$?

Solution. To check and see if the three vectors are linearly dependent, we have

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 9 \\ -1 \\ -2 \end{pmatrix} = 0.$$

Using the augmented matrix form, we have

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & -3 & 9 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \sim \begin{matrix} \text{d} = \text{a} + 3\text{c} \\ \text{e} = \text{a} - \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \sim \begin{matrix} \text{d} \\ \text{c} \\ \text{e} + 5\text{c} \end{matrix} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence c_3 is free, so the three vectors are linearly dependent. Therefore, $\dim(\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}) = 2$.

(b) Is the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ linearly independent? Justify your answer.

Solution. No, since the subset $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly dependent. (Alternatively, we know that any set of four vectors in $\mathcal{R}^3$ must be linearly dependent.)

(c) Identify which subsets of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ form bases of $\mathcal{R}^3$.

Solution. Any such subset must have three elements. $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ does not qualify since the vectors are linearly dependent. However, any two of the vectors are linearly independent. Therefore, we must only check to see if $\mathbf{x}_4$ depends on any two of these vectors. Therefore, we must solve

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Using the augmented matrix form, we have

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & -3 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{matrix} \text{d} = \text{a} + 3\text{c} \\ \text{e} = \text{a} - \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{matrix} \text{d} \\ \text{e} \\ \text{e} + 5\text{f} \end{matrix} \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 6 & 0 \end{pmatrix},$$

so the only solution is $\mathbf{0}$, and hence the vectors are linearly independent. Therefore, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$, $\{\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4\}$, and $\{\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ all form bases for $\mathcal{R}^3$.

6. Find the dimension of and a basis for each of the vector spaces listed.

$$\text{(a)} : \quad \begin{pmatrix} 2a - b \\ a + b \\ b \\ -a + 3b \end{pmatrix}, \qquad a, b \in \mathcal{R}$$

Solution.

$$\begin{pmatrix} 2a-b \\ a+b \\ b \\ -a+3b \end{pmatrix} = a\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2.$$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of one another, so they are linearly independent. Therefore, they form a basis for the vector space, which has dimension 2.

$$\text{(b)}: \quad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \qquad a, b \in \mathcal{R}$$

Solution.

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = aM_1 + bM_2.$$

M_1 and M_2 are not multiples of one another, so they are linearly independent. Therefore, they form a basis for the vector space, which has dimension 2.

7. For each of the following matrices, find a basis for row A, col A, and $\mathcal{N}(A)$.

(a)

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{pmatrix}$$

Solution. We begin by finding $\mathcal{N}(A)$ by writing the system in augmented matrix form:

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix}\begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 4 & 7 & 8 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} \\ \text{d} = \text{b} - 2\text{a} \\ \text{e} = 4\text{a} - \text{c} \end{matrix}\begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} + 3\text{d}/5 \\ -\text{d}/5 \\ \text{d} + \text{e} \end{matrix}\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solution is given by $(-2x_3, 0, x_3)^T$, and hence a basis for $\mathcal{N}(A)$ is given by $(-2, 0, 1)^T$. We note the dimension of the column space and the row space must then be 2. The pivot columns are 1 and 2, so a basis for col A is given by the first and second columns of A. No row is a multiple of another row, so a basis for row A is given by any two of the three rows, or by the nonzero rows $(1, 0, 2)$ and $(0, 1, 0)$ of the reduced echelon form. (Other choices for the bases are possible.)

(b)

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}$$

Solution. We begin by finding $\mathcal{N}(A)$ by writing the system in augmented matrix form:

$$\begin{matrix}\text{a}\\ \text{b}\\ \text{c}\end{matrix}\begin{pmatrix}-3 & 1 & 3 & 4 & 0\\ 1 & 2 & -1 & -2 & 0\\ -3 & 8 & 4 & 2 & 0\end{pmatrix} \sim \begin{matrix}\text{b}\\ \text{d} = \text{a}+3\text{b}\\ \text{e}=\text{a}-\text{c}\end{matrix}\begin{pmatrix}1 & 2 & -1 & -2 & 0\\ 0 & 7 & 0 & -2 & 0\\ 0 & -7 & -1 & 2 & 0\end{pmatrix}$$

$$\sim \begin{matrix}\text{f}=7\text{b}+2\text{e}\\ \text{d}\\ \text{g}=\text{d}+\text{e}\end{matrix}\begin{pmatrix}7 & 0 & -9 & -10 & 0\\ 0 & 7 & 0 & -2 & 0\\ 0 & 0 & -1 & 0 & 0\end{pmatrix}$$

$$\sim \begin{matrix}\text{f}-9\text{g}\\ \text{d}\\ \text{g}\end{matrix}\begin{pmatrix}7 & 0 & 0 & -10 & 0\\ 0 & 7 & 0 & -2 & 0\\ 0 & 0 & -1 & 0 & 0\end{pmatrix}.$$

Therefore, the solution is given by $(10x_4/7, 2x_4/7, 0, x_4)^T$, and hence a basis for $\mathcal{N}(A)$ is given by $(10/7, 2/7, 0, 1)^T$. We note the dimension of the column space and the row space must then be 3. Therefore, the rows of A form a basis for row A. Alternatively, we may use the nonzero rows $(7,0,0,-10)$, $(0,7,0,-2)$, and $(0,0,-1,0)$ of the reduced echelon form. Using the rule of thumb that the pivot columns correspond to basis vectors, we see that the first three columns of A may serve as a basis for $\operatorname{col} A$. (Other choices for the bases are possible.)

8. Consider the following matrix and vector:

$$A = \begin{pmatrix}3 & -1\\ -6 & 2\\ -3 & 1\end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix}-4\\ 8\\ 4\end{pmatrix}.$$

(a) Is $\mathbf{b} \in \operatorname{col} A$?

Solution. $\mathbf{b} \in \operatorname{col} A$ implies that $A\mathbf{x} = \mathbf{b}$ has a solution. Writing this in augmented matrix form, we have

$$\begin{matrix}\text{a}\\ \text{b}\\ \text{c}\end{matrix}\begin{pmatrix}3 & -1 & -4\\ -6 & 2 & 8\\ -3 & 1 & 4\end{pmatrix} \sim \begin{matrix}\text{a}\\ \text{b}+2\text{a}\\ \text{c}+\text{a}\end{matrix}\begin{pmatrix}3 & -1 & -4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Therefore, the solution is given by

$$\mathbf{x} = \begin{pmatrix}(x_2-4)/3\\ x_2\end{pmatrix},$$

and hence $\mathbf{b} \in \operatorname{col} A$.

(b) Is the system $A\mathbf{x} = \mathbf{b}$ consistent? If so, how many solutions are there?

Solution. From part (a) we see that the system is consistent and has a a one-dimensional family of solutions.

9. Let $B = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n]$ and $F = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$ be two ordered bases for $\mathcal{R}^n$ and let $U = (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n)$ and $V = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ be two $n \times n$ matrices whose columns are the basis vectors in B and F, respectively. Show that the transition matrix ${}_{B\leftarrow F}P$ from F to B can be determined by calculating the reduced row echelon form of $(U|V)$.

Solution. By notes in class, we know that

$$U = {}_{E\leftarrow B}P, \qquad V = {}_{E\leftarrow F}P \qquad \Longrightarrow \qquad U^{-1} = {}_{B\leftarrow E}P$$

$${}_{B\leftarrow F}P = {}_{B\leftarrow E}P\,{}_{E\leftarrow F}P = U^{-1}V$$

However, since U has linearly independent columns, $(U|V)$ has rank n, and hence its reduced row echelon form is $(I|U^{-1}V)$. Therefore, the right half of the augmented matrix is the desired transition matrix.

10. Consider the two ordered bases $B = [1, 1+x, 1+x+x^2]$ and $C = [1, 1-x, 1-x^2]$ for $\mathcal{P}_2$.

(a) Find the transition matrix ${}_{C\leftarrow B}P$.

Solution. The first column of ${}_{C\leftarrow B}P$ is given by $[\mathbf{b}_1]_C = [1]_C$. Therefore, we have to find the solution of the following equation:

$$1 = c_1(1) + c_2(1-x) + c_3(1-x^2).$$

There is no power of x^2 on the left-hand side, so $c_3 = 0$. There is no power of x on the left-hand side, so that means that $c_2 = 0$. So we have $c_1 = 1$, and $[\mathbf{b}_1]_C = (1, 0, 0)^T$. Calculating $[\mathbf{b}_2]_C$, we have

$$1 + x = c_1(1) + c_2(1-x) + c_3(1-x^2).$$

There is no power of x^2 on the left-hand side, so $c_3 = 0$. Then matching the x terms, we have $1 = -c_2$, so $c_2 = -1$. Then the constant terms become

$$\begin{aligned} 1 &= c_1 + c_2 + c_3 = c_1 - 1 \\ c_1 &= 2 \qquad \Longrightarrow \qquad [\mathbf{b}_2]_C = (2, -1, 0)^T. \end{aligned}$$

Calculating $[\mathbf{b}_3]_C$, we have

$$1 + x + x^2 = c_1(1) + c_2(1-x) + c_3(1-x^2).$$

Matching the x^2 terms, we have $-1 = c_3$. Then matching the x terms, we have $1 = -c_2$, so $c_2 = -1$. Then the constant terms become

$$\begin{aligned} 1 &= c_1 + c_2 + c_3 = c_1 - 2 \\ c_1 &= 3 \qquad \Longrightarrow \qquad [\mathbf{b}_3]_C = (3, -1, -1)^T. \end{aligned}$$

Then slotting these three results into the columns of ${}_{C\leftarrow}P_B$, we have

$$ {}_{C\leftarrow}P_B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}. $$

(b) Calculate $\mathbf{b} = [2x^2 + x]_B$ and $\mathbf{c} = [2x^2 + x]_C$.

Solution. To find $\mathbf{b}$, we are trying to find the solution of the following equation:

$$ 2x^2 + x = b_1(1) + b_2(1 + x) + b_3(1 + x + x^2). $$

Matching coefficients of powers of x^2, we have that $b_3 = 2$. Then matching the x and constant terms, we have

$$ \begin{aligned} 1 &= b_2 + b_3 = b_2 + 2 \\ b_2 &= -1, \\ 0 &= b_1 + b_2 + b_3 = b_1 + 1 \\ b_1 &= -1 \qquad \Longrightarrow \qquad \mathbf{b} = (-1, -1, 2)^T. \end{aligned} $$

To find $\mathbf{c}$, we are trying to find the solution of the following equation:

$$ 2x^2 + x = c_1(1) + c_2(1 - x) + c_3(1 - x^2). $$

Matching coefficients of powers of x^2, we have that $c_3 = -2$. Then matching powers of x, we have $1 = -c_2$, $c_2 = -1$. Then the constant terms become

$$ \begin{aligned} 0 &= c_1 + c_2 + c_3 = c_1 - 3 \\ c_1 &= 3 \qquad \Longrightarrow \qquad \mathbf{c} = (3, -1, -2)^T. \end{aligned} $$

(c) Verify that $\mathbf{c} = {}_{C\leftarrow}P_B\,\mathbf{b}$.

Solution. Using our answers to (a) and (b), we have

$$ {}_{C\leftarrow}P_B\,\mathbf{b} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \mathbf{c}. $$