MATH 351-010 Engineering Mathematics I
Prof. D. A. Edwards Due: Oct. 28, 2022

Homework Set 7 Solutions

1. If B is invertible, prove that $\det(B^{-1}AB) = \det A$.

Solution.

$$\det(B^{-1}AB) = (\det B^{-1})(\det A)(\det B) = (\det B)^{-1}(\det A)(\det B) = \det A.$$

2. Calculate $\det B$, where

$$B = \begin{pmatrix} 0 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 2 & 2 & 1 \\ -1 & -1 & 3 & 7 \end{pmatrix}.$$

Solution. Expanding by the second row, we have

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 2 & 2 & 1 \\ -1 & -1 & 3 & 7 \end{vmatrix} = -4 \begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 7 \end{vmatrix} = (-4)(-2) \begin{vmatrix} 1 & 1 \\ -1 & 7 \end{vmatrix} = 8(7+1) = 64,$$

where in the second step we expanded by the first row.

3. Consider the following matrix in $\mathcal{R}^{5\times 5}$:

$$A = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix},$$

where the $*$s are arbitrary entries. Show that $\det A = 0$.

Solution. We insert some numbers for our entries where needed:

$$A = \begin{pmatrix} a_{11} & a_{12} & * & * & * \\ a_{21} & a_{22} & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}.$$

Then expanding by the first column, we have

$$\det A = a_{11}\begin{vmatrix} a_{22} & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{vmatrix}$$

$$\det A = a_{11}a_{22}\begin{vmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{vmatrix} - a_{21}a_{12}\begin{vmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{vmatrix} = 0.$$

4.

(a) Calculate the determinant of

$$\begin{pmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{pmatrix}.$$

Solution.

$$\begin{vmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - (3)(5) = -12 - 4\lambda + \lambda^2.$$

(b) Find the value(s) of λ such that the determinant is 0.

Solution.

$$-12 - 4\lambda + \lambda^2 = (\lambda - 6)(\lambda + 2) = 0 \qquad \Longrightarrow \qquad \lambda = -2, 6$$

5. Let $A \in \mathcal{R}^{n\times n}$, and let the first and second rows of A be equal.

(a) Calculate the determinant by expanding along each of the first and second rows.

Solution. Expanding along the first row, we have

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}. \tag{A.1}$$

Expanding along the second row, we have

$$\det A = \sum_{j=1}^{n} (-1)^{2+j} a_{2j} \det A_{2j}. \tag{A.2}$$

(b) Show that $\det A = 0$.

Solution. Since the first and second rows of A are equivalent, we see that $\det A_{2j} = \det A_{1j}$ and $a_{1j} = a_{2j}$. Therefore, (A.2) becomes

$$\det A = \sum_{j=1}^{n} (-1)^{2+j} a_{1j} \det A_{1j} = -\sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j},$$

where we have factored out the minus sign. But now we just have minus the right-hand side of (A.1), so we have that $\det A = -\det A$. Hence $\det A = 0$.

6. Suppose that A, B, and C are $n \times n$ matrices such that $\det A = 2$, $\det B = -5/3$, and $\det C = 0$. Calculate the following determinants:

   (a) $\det A^{-1}$

Solution. $\det A^{-1} = (\det A)^{-1} = 1/2$.

   (b) $\det(3B)$

Solution. $\det(3B) = 3^n \det B = -5 \cdot 3^{n-1}$.

   (c) $\det(ABC)$

Solution. $\det(ABC) = (\det A)(\det B)(\det C) = (2)(-5/3)(0) = 0$.

   (d) $\det(A^T B^{-1})$

Solution. $\det(A^T B^{-1}) = (\det A)(\det B)^{-1} = 2(-3/5) = -6/5$.

7. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

   Show by direct calculation that $\det AB = (\det A)(\det B)$.

Solution.

$$\det A = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 - (-2) = 5, \qquad B = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4 - 6 = -2,$$

$$\det AB = \begin{vmatrix} 8 & 5 \\ 2 & 0 \end{vmatrix} = 0 - 10 = -10 = (\det A)(\det B).$$

8. Determine whether each of the following sets is linearly independent in the appropriate vector space:

   (a) $$\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix} \right\}$$

Solution. One way to check for linear independence is to solve $\sum_{i=1}^{3} c_i \mathbf{v}_i = \mathbf{0}$ and see what the c_i are. Solving the augmented matrix system, we have

$$\begin{matrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{matrix} \begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 5 & 0 \\ -1 & 4 & 11 & 0 \end{pmatrix} \sim \begin{matrix} \mathrm{d} = \mathrm{a}/2 \\ \mathrm{e} = \mathrm{a} - 2\mathrm{b} \\ \mathrm{f} = \mathrm{a} + \mathrm{c} \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 6 & 16 & 0 \end{pmatrix} \sim \begin{matrix} \mathrm{d}/4 \\ 2\mathrm{e} + \mathrm{f} \\ 3\mathrm{e} + 2\mathrm{f} \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 8 & 0 \end{pmatrix}.$$

Therefore, the only solution is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, so the set is linearly independent.

(b) $$\left\{\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}\right\}$$

Solution. One way to check for linear independence is to see if you can write one of the vectors as a linear combination of the others. In this case, that is equivalent to

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} = c\begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}$$

But there is no value of c which makes the upper left entries agree. So the set is linearly independent.

(c) $$\left\{\sin^3 x, \sin 3x, \sin x\right\}$$

Solution. Using the trigonometric identity that

$$\sin^3 x = \frac{-\sin 3x + 3\sin x}{4},$$

we see that the first vector is a linear combination of the second two, so this set is linearly dependent.

9. (BH) Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in a vector space V. Prove that $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is also linearly independent, where $\mathbf{w}_1 = \mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_3$.

Solution. If T is a linearly independent set, we must have that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$

implies that all the $c_i = 0$. Using the definition of the $\mathbf{w}_i$, we have

$$\begin{aligned} c_1(\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3) + c_2(\mathbf{v}_2 - \mathbf{v}_3) + c_3\mathbf{v}_3 &= \mathbf{0} \\ c_1\mathbf{v}_1 + (c_2 - c_1)\mathbf{v}_2 + (2c_1 - c_2 + c_3)\mathbf{v}_3 &= \mathbf{0}. \end{aligned}$$

Since S is a linearly independent set, we know that the last equation is satisfied only when each of the coefficients of the $\mathbf{v}_i$ is zero. Therefore, we have that

$$c_1 = 0, \qquad c_2 - c_1 = 0, \qquad 2c_1 - c_2 + c_3 = 0.$$

Substituting the first result into the second equation give us that $c_2 = 0$. Substituting these two results into the last equation gives us that $c_3 = 0$. Therefore, all the c_i must be 0, and the result is proved.

10. (BH) Consider the following vectors:

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -3 \\ z \\ 9 \end{pmatrix}.$$

(a) For which value(s) of z does the equation $\mathbf{x}_3 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ have a solution?

Solution. We wish to solve the following system:

$$\begin{pmatrix} -3 \\ z \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} 2c_1 - c_2 &= -3 \\ 3c_1 + 2c_2 &= z \\ -6c_1 + 3c_2 &= 9. \end{aligned}$$

We note that the last equation is -3 times the first, so when the first is satisfied the last is automatically satisfied. Therefore, we really have only two equations in c_1 and c_2, which will have a solution for all z (since they aren't multiples of each other).

(b) For which value(s) of z are $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly dependent?

Solution. Since $\mathbf{x}_3$ can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ for all z, the set is linearly dependent for all z.