Homework Set 6 Solutions

1. Consider the following matrices:

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 3 \\ 2 & -3 \\ -4 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix}.$$

If possible, compute

(a) $A(BC)$

Solution.

$$A(BC) = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 2 \end{pmatrix} \left[\begin{pmatrix} 0 & 3 \\ 2 & -3 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix} \right] = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 18 & -9 \\ -14 & 7 \\ -14 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & -4 \\ 4 & -2 \end{pmatrix}.$$

(b) $(AB)C$

Solution.

$$(AB)C = \left[\begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 2 & -3 \\ -4 & -1 \end{pmatrix} \right] \begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix} = \begin{pmatrix} -14 & 6 \\ -10 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & -4 \\ 4 & -2 \end{pmatrix}.$$

2. Consider the following matrices:

$$D = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -2 & 3 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ -3 & 1 & -2 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 3 & -3 \\ 2 & 1 & 4 \end{pmatrix}.$$

If possible, compute

(a) $D(E+F)$

Solution.

$$D(E+F) = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -2 & 3 \end{pmatrix} \left[\begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ -3 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ -3 & 3 & -3 \\ 2 & 1 & 4 \end{pmatrix} \right]$$
$$= \begin{pmatrix} 1 & -2 & 1 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ -4 & 5 & -2 \\ -1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -5 & 14 \\ 9 & 2 & 18 \end{pmatrix}.$$

Copyright ©2022 D. A. Edwards All Rights Reserved

(b) $DE + DF$

Solution.

$$DE + DF = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ -3 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -3 & 3 & -3 \\ 2 & 1 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 2 \\ -5 & 5 & -4 \end{pmatrix} + \begin{pmatrix} 9 & -5 & 12 \\ 14 & -3 & 22 \end{pmatrix} = \begin{pmatrix} 9 & -5 & 14 \\ 9 & 2 & 18 \end{pmatrix}.$$

3. Let $A \in \mathcal{R}^{n\times n}$ and let $B = A + A^T$, $C = A - A^T$.

(a) Show that B is symmetric and C is antisymmetric.

Solution.

$$B^T = (A + A^T)^T = A^T + A = B$$
$$C^T = (A - A^T)^T = A^T - A = -C.$$

(b) Show that every $n \times n$ matrix can be represented as a sum of a symmetric matrix and an antisymmetric matrix.

Solution. Let A be that matrix. Then

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2} = \frac{B}{2} + \frac{C}{2}.$$

Since multiplying a matrix by a constant doesn't change its symmetry, the result is proven.

4. Consider the linear system

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 3, \\ 4x_1 - x_2 - x_3 &= 3, \\ -x_1 + 7x_2 + x_3 &= 0. \end{aligned} \tag{6.1}$$

(a) Write the system (6.1) as a matrix-vector equation.

Solution. The system may be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & -1 & -1 \\ -1 & 7 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

(b) Solve the system (6.1).

Solution. Using the augmented matrix notation, we have

$$\begin{matrix}\text{a}\\\text{b}\\\text{c}\end{matrix}\begin{pmatrix}1 & 3 & 2 & 3\\4 & -1 & -1 & 3\\-1 & 7 & 1 & 0\end{pmatrix} \sim \begin{matrix}\text{a}\\\text{d}=\text{b}+4\text{c}\\\text{e}=\text{a}+\text{c}\end{matrix}\begin{pmatrix}1 & 3 & 2 & 3\\0 & 27 & 3 & 3\\0 & 10 & 3 & 3\end{pmatrix}$$

$$\sim \begin{matrix}\text{a}\\\text{f}=\text{d}-\text{e}\\\text{g}=\text{e}-10\text{f}/17\end{matrix}\begin{pmatrix}1 & 3 & 2 & 3\\0 & 17 & 0 & 0\\0 & 0 & 3 & 3\end{pmatrix}$$

$$\sim \begin{matrix}\text{a}-3\text{h}-2\text{j}\\\text{h}=\text{f}/17\\\text{j}=\text{g}/3\end{matrix}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 1\end{pmatrix},$$

so the solution is $\mathbf{x} = (1, 0, 1)^T$.

5. Consider the following matrices:

$$A = \begin{pmatrix}4 & 1 & 6\\2 & 3 & 5\end{pmatrix}, \qquad \begin{pmatrix}1 & 3 & 0\\-2 & 2 & -4\end{pmatrix}.$$

(a) Verify that $A + B = B + A$, $3(A + B) = 3A + 3B$, and $(A + B)^T = A^T + B^T$.

Solution.

$$\begin{aligned}
A + B &= \begin{pmatrix}4 & 1 & 6\\2 & 3 & 5\end{pmatrix} + \begin{pmatrix}1 & 3 & 0\\-2 & 2 & -4\end{pmatrix} = \begin{pmatrix}5 & 4 & 6\\0 & 5 & 1\end{pmatrix}\\
B + A &= \begin{pmatrix}1 & 3 & 0\\-2 & 2 & -4\end{pmatrix} + \begin{pmatrix}4 & 1 & 6\\2 & 3 & 5\end{pmatrix} = \begin{pmatrix}5 & 4 & 6\\0 & 5 & 1\end{pmatrix},\\
3(A + B) &= 3\begin{pmatrix}5 & 4 & 6\\0 & 5 & 1\end{pmatrix} = \begin{pmatrix}15 & 12 & 18\\0 & 15 & 3\end{pmatrix}\\
3A + 3B &= 3\begin{pmatrix}4 & 1 & 6\\2 & 3 & 5\end{pmatrix} + 3\begin{pmatrix}1 & 3 & 0\\-2 & 2 & -4\end{pmatrix} = \begin{pmatrix}12 & 3 & 18\\6 & 9 & 15\end{pmatrix} + \begin{pmatrix}3 & 9 & 0\\-6 & 6 & -12\end{pmatrix}\\
&= \begin{pmatrix}15 & 12 & 18\\0 & 15 & 3\end{pmatrix},\\
(A + B)^T &= \begin{pmatrix}5 & 4 & 6\\0 & 5 & 1\end{pmatrix}^T = \begin{pmatrix}5 & 0\\4 & 5\\6 & 1\end{pmatrix}\\
A^T + B^T &= \begin{pmatrix}4 & 2\\1 & 3\\6 & 5\end{pmatrix} + \begin{pmatrix}1 & -2\\3 & 2\\0 & -4\end{pmatrix} = \begin{pmatrix}5 & 0\\4 & 5\\6 & 1\end{pmatrix}.
\end{aligned}$$

(b) Calculate the following (or indicate the product doesn't exist):

$$AB, \qquad A^T B, \qquad A^T B^T.$$

Solution. AB doesn't exist because the number of columns of A is not the same as the number of rows of B. $A^T B^T$ doesn't exist because the number of rows of A is not the same as the number of columns of B.

$$A^T B = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 16 & -8 \\ -5 & 9 & -12 \\ -4 & 28 & -20 \end{pmatrix}.$$

6. Let

$$A = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

(a) Compute $(AB^T)C$.

Solution.

$$(AB^T)C = \left[\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \right] \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ 3 & -6 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -15 \end{pmatrix}.$$

(b) Compute $B^T C$ and multiply the result by A on the right.

Solution.

$$(B^T C)A = \left[\begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -15 \end{pmatrix}.$$

(c) Explain why $(AB^T)C = (B^T C)A$.

Solution. By the associative property for matrix multiplication, $(AB^T)C = A(B^T C)$. But $B^T C$ is a scalar, so it commutes with any matrix, so $A(B^T C) = (B^T C)A$, and the result is proved.

7. Show that if A is a symmetric nonsingular matrix, then A^{-1} is also symmetric.

Solution.

$$\begin{aligned} AA^{-1} &= I \\ (AA^{-1})^T &= I^T = I \\ A^{-T} A^T = A^{-T} A &= I, \end{aligned}$$

where in the last line we have used the fact that A is symmetric. But by the definition of the inverse, $A^{-T} = A^{-1}$, so A^{-1} is symmetric too.

8. Let

$$C = \begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}.$$

(a) Calculate C^{-1}.

Solution. Using the augmented matrix notation, we have

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} -1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \sim \begin{matrix} \text{d} = -\text{a} \\ \text{e} = 2\text{a} + \text{b} \\ \text{f} = \text{a} + \text{c} \end{matrix} \begin{pmatrix} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 6 & 2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{matrix} \text{d} + 2\text{g} \\ 6\text{f} - 5\text{e} \\ \text{g} = \text{e} - \text{f} \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -4 & -5 & 6 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

Therefore, we have that

$$C^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ -4 & -5 & 6 \\ 1 & 1 & -1 \end{pmatrix}. \tag{A}$$

(b) Use your answer to (a) to show that the solution of $C\mathbf{x} = \mathbf{b}$ for the following cases is the vector listed:

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \implies \mathbf{x} = \begin{pmatrix} 2 \\ -8 \\ 2 \end{pmatrix}.$$

Solution. In each case we have that $\mathbf{x} = C^{-1}\mathbf{b}$, so using (A), we have in each case

$$\mathbf{x} = \begin{pmatrix} 1 & 2 & -2 \\ -4 & -5 & 6 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} 1 & 2 & -2 \\ -4 & -5 & 6 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 2 \end{pmatrix}.$$

9. For each pair of matrices $\{A, B\}$, find an elementary matrix E such that $EA = B$. Also, explain in words what row operations each elementary matrix performs.

$$A_1 = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}, \tag{a}$$

$$A_2 = \begin{pmatrix} 2 & -3 \\ 3 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 2 & -3 \\ -9 & -3 \end{pmatrix}, \tag{b}$$

$$A_3 = \begin{pmatrix} 2 & 4 & 7 \\ -1 & 4 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 2 & 4 & 7 \\ 0 & 9 & 5 \\ 2 & 1 & 1 \end{pmatrix}. \tag{c}$$

Solution. To obtain B_1 from A_1, we interchange the rows, so we obtain

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To obtain B_2 from A_2, we multiply the second row by -3. The first row remains unchanged, so we obtain

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}.$$

To obtain B_3 from A_3, we must add twice the second row to the third row to obtain the new second row. The first and third rows remain unchanged, so we obtain

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

10. Consider the following matrix:

$$C = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 1 & \alpha & 1 & \alpha \\ 1 & -1 & \alpha & 1 \end{pmatrix}.$$

Show that C is not invertible *only if* $\alpha = \pm 1$.

Solution. Beginning to row reduce C, we note that

$$\begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{array} \begin{pmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 1 & \alpha & 1 & \alpha \\ 1 & -1 & \alpha & 1 \end{pmatrix} \sim \begin{array}{l} \mathrm{e} = \mathrm{a} - \alpha \mathrm{b} \\ \mathrm{b} \\ \mathrm{f} = \mathrm{c} - \alpha \mathrm{d} \\ \mathrm{d} \end{array} \begin{pmatrix} 1-\alpha^2 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 1-\alpha & 2\alpha & 1-\alpha^2 & 0 \\ 1 & -1 & \alpha & 1 \end{pmatrix}.$$

Clearly, e is a row of zeroes if $\alpha^2 = 1$, or $\alpha = \pm 1$. Now suppose we continued to row-reduce. There's no way to make any other row a row of zeroes, since using the rows above can't zero out the diagonal entry, and using the rows below will make the entries to the right of the diagonal entry nonzero. So we get a row of zeroes only if $\alpha = \pm 1$. In that case C is not row equivalent to I, and hence is not invertible.