Homework Set 5 Solutions

1. Consider the equation

$$y'' + y = \csc^2 x.$$

(a) Solve for a particular solution using the method of variation of parameters.

Solution. The homogeneous solutions are trivially given by

$$y_1 = \cos x, \qquad y_2 = \sin x$$

so the Wronskian is given by

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (\cos^2 x + \sin^2 x) = 1.$$

Using the variation of parameters formula, we have

$$\begin{aligned} y_{\mathrm{p}}(x) &= -\cos x \int \csc^2 x \sin x \, dx + \sin x \int \csc^2 x \cos x \, dx \\ &= -\cos x \int \csc x \, dx + \sin x \int \frac{\cos x}{\sin^2 x} \, dx \\ &= -\cos x \log|\csc x - \cot x| + \sin x \int \frac{du}{u^2}, \quad u = \sin x \\ &= -\cos x \log|\csc x - \cot x| - \frac{\sin x}{u} \\ &= -\cos x \log|\csc x - \cot x| - 1, \end{aligned}$$

(b) Where is this equation guaranteed to have a unique solution?

Solution. $\csc^2 x$ is undefined whenever $\sin x = 0$, or when $x = n\pi$, n an integer. So the solution is guaranteed to have a unique solution in any interval *not* containing $x = n\pi$.

2. Find the general solution of

$$\ddot{y} - 4\dot{y} + 4y = \frac{e^{2t}}{t}.$$

Solution. We begin by finding the homogeneous solution by substituting in $e^{\lambda t}$ to the homogeneous form, which yields

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

Since we have a repeated root, the homogeneous solution is given by

$$y_{\mathrm{h}}(t) = e^{2t}(c_1 + c_2 t).$$

Then the Wronskian is given by

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (1+2t)e^{2t} \end{vmatrix} = (1+2t)e^{4t} - (2te^{4t}) = e^{4t}.$$

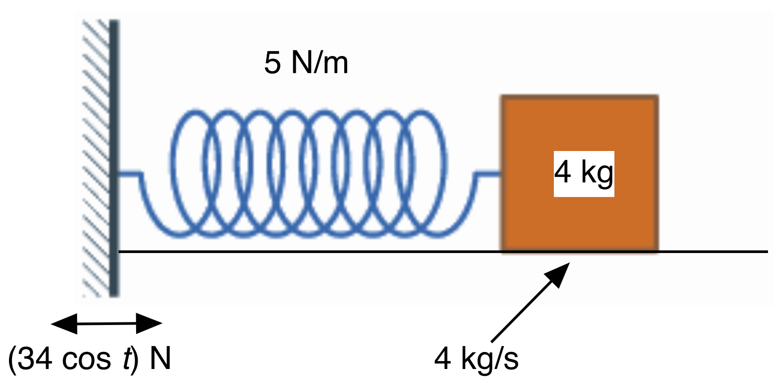
Using the variation of parameters formula, we have that the particular solution is given by

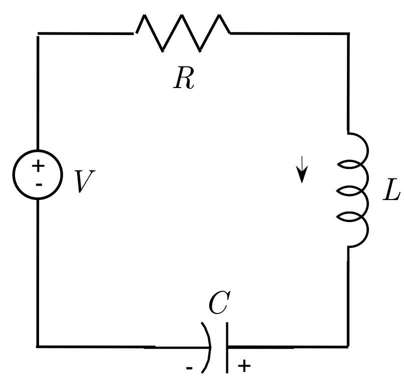
$$\begin{aligned} y_{\mathrm{p}}(t) &= -e^{2t}\int \frac{te^{2t}}{e^{4t}}\frac{e^{2t}}{t}\,dt + te^{2t}\int \frac{e^{2t}}{e^{4t}}\frac{e^{2t}}{t} \\ &= -e^{2t}\int dt + te^{2t}\int \frac{dt}{t} \\ &= -e^{2t}t + te^{2t}\log t. \end{aligned}$$

Thus the general solution is given by the homogenous solution plus the particular solution:

$$y(t) = te^{2t}\log t + (c_1 + c_2 t)e^{2t},$$

where we have folded the e^{2t} term in the particular solution into the arbitrary constant A.





3. We reconsider the spring system of Homework Set 3, #5. As before, the spring has stiffness $k = 5$ N/m, is damped with damping constant $b = 4$ kg/s, and is attached to a weight with mass $M = 4$ kg. However, now we impose the following discontinuous oscillation of the support (in Newtons; see figure):

$$F(t) = \begin{cases} 34\cos t, & 0 \le t \le 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

Initially the spring is extended 2 m and given a velocity of -2 m/s.

(a) Solve the resulting system for the displacement $x(t)$ in the region $t \in [0, 2\pi]$.

Solution. From homework set 3, we know that the left-hand side of the equation is $4\ddot{x} + 4\dot{x} + 5x$. Then equating this to the additional force given on the support, we have

$$4\ddot{x} + 4\dot{x} + 5x = 34\cos t, \qquad t \in [0, 2\pi]; \qquad x(0) = 2, \quad \dot{x}(0) = -2.$$

Using the method of undetermined coefficients, we substitute

$$x_{\mathrm{p}} = p_1 \cos t + p_2 \sin t$$

into our equation to obtain

$$\begin{aligned} -4(p_1 \cos t + p_2 \sin t) + 4(-p_1 \sin t + p_2 \cos t) + 5(p_1 \cos t + p_2 \sin t) &= 34\cos t \\ (p_1 + 4p_2)\cos t + (p_2 - 4p_1)\sin t &= 34 \cos t \\ p_1 = 2, \qquad p_2 &= 8. \end{aligned}$$

Using our answer to homework set 3, we have that our full solution is given by

$$x(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t) + 2(\cos t + 4\sin t).$$

Solving the first initial condition, we immediately have that $c_1 = 0$. Solving the second initial condition, we have

$$\begin{aligned} \dot{x}(0) = e^{-t/2}(c_2 \cos t) - \frac{e^{-t/2}(c_2 \sin t)}{2} + 8\bigg|_{t=0} &= -2 \\ c_2 &= -10 \end{aligned}$$

$$x(t) = -10e^{-t/2}\sin t + 2(\cos t + 4\sin t). \tag{A}$$

(b) Show that

$$x(2\pi) = 2, \qquad \dot{x}(2\pi) = 8 - 10e^{-\pi}. \tag{5.1}$$

Solution. Substituting $t = 2\pi$ into (A), we obtain

$$x(2\pi) = 2,$$

as required. Taking the derivative of (A) and substituting $t = 2\pi$, we obtain

$$\begin{aligned} \dot{x}(2\pi) &= e^{-t/2}(-10\cos t) - \frac{e^{-t/2}(-10\sin t)}{2} + 8\cos t\bigg|_{t=2\pi} \\ &= -10e^{-\pi} + 8. \end{aligned}$$

(c) Using the fact that x and $\dot{x}$ should be continuous at $t = 2\pi$, calculate the displacement x in the region $t > 2\pi$.

Solution. Since the forcing is now zero, this is exactly Homework Set 3, #5, so our solution is given by

$$x(t) = e^{-(t-2\pi)/2}(d_1 \cos t + d_2 \sin t),$$

where we have replaced t by $t - 2\pi$ for algebraic convenience. (This is equivalent to multiplying by a constant.) Substituting this expression into (5.1) to obtain the constants, we have that

$$x(2\pi) = d_1 = 2.$$

Taking the derivative of (F) and substituting $t = 2\pi$, we obtain

$$\begin{aligned} \dot{x}(2\pi) &= e^{-(t-2\pi)/2}(-2\sin t + d_2 \cos t) - \left.\frac{e^{-(t-2\pi)/2}(2\cos t + d_2 \sin t)}{2}\right|_{t=2\pi} \\ -10e^{-\pi} + 8 &= d_2 - 1 \\ d_2 &= 9 - 10e^{-\pi} \end{aligned}$$

$$x(t) = e^{-(t-2\pi)/2}\left[2\cos t + (9 - 10e^{-\pi})\sin t\right].$$

4. We reconsider the *series RLC* circuit of Homework Set 4, #4, but now we apply an AC voltage of $B\cos\omega t$ (see figure above right). There is an initial voltage on the capacitor of V_0 and initially there is a current I_0 in the inductor.

   (a) Explain in words why the governing equations for this system are

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = -B\omega\sin\omega t, \qquad I(0) = I_0. \tag{5.2}$$

Solution. From notes in class, we have that balance of voltage around the loop yields

$$L\dot{I} + RI + \frac{1}{C}\int_0^t I + V_0 = B\cos\omega t. \tag{B}$$

Taking the derivative of this equation with respect to t yields the first equation in (5.2). The initial condition for the current is given in the problem statement.

   (b) Show that in the case where $L = 1$, $C = 1/5$, and $R = 4$, the steady state for the system is

$$I_s(t) = \frac{B\omega[4\omega\cos\omega t + (\omega^2 - 5)\sin\omega t]}{\omega^4 + 6\omega^2 + 25}. \tag{5.3}$$

Solution. From Homework Set 4, #4, we know that the homogeneous solutions decay to zero as $t \to \infty$. Therefore, the steady state is given by the particular solution. Using the method of undetermined coefficients, we substitute

$$I_s(t) = p_1 \cos\omega t + p_2 \sin\omega t \tag{C}$$

into (5.2) with $L = 1$, $C = 1/5$, and $R = 4$ to obtain

$$-\omega^2(p_1\cos\omega t + p_2\sin\omega t) + 3\omega(-p_1\sin\omega t + p_2\cos\omega t) + 5(p_1\cos\omega t + p_2\sin\omega t) \\ = -B\omega\sin\omega t$$

$$\big[(-\omega^2+5)p_1+4\omega p_2\big]\cos\omega t+\big[(-\omega^2+5)p_2-4\omega p_1\big]\sin\omega t=-B\omega\sin\omega t.$$

Matching coefficients of sine and cosine, we have

$$\begin{aligned}
4\omega p_1+(\omega^2-5)p_2&=B\omega\\
(\omega^2-5)p_1-4\omega p_2&=0\\
p_1=\frac{4B\omega^2}{(4\omega)^2+(\omega^2-5)^2}&=\frac{4B\omega^2}{\omega^4+6\omega^2+25},\\
p_2=\frac{B\omega(\omega^2-5)}{(4\omega)^2+(\omega^2-5)^2}&=\frac{B\omega(\omega^2-5)}{\omega^4+6\omega^2+25}.
\end{aligned}$$

Substituting p_1 and p_2 into (C) and simplifying, we obtain (5.3).

5. An oil refinery produces low-sulfur and high-sulfur fuel.
   - Producing one ton of low-sulfur fuel takes 10 minutes in the blending plant and 8 minutes in the refining plant.
   - Producing one ton of high-sulfur fuel takes 8 minutes in the blending plant and 4 minutes in the refining plant.
   - The blending plant is available for 8 hours a day, and the refining plant is available for 6 hours a day.

   (a) Write the system of equations needed to solve for the tons of l and h necessary to utilize the plant fully.

Solution. Balancing the amount of time using minutes, we have

$$\begin{aligned}
10l+8h&=480 &&\text{(blending plant)}\\
8l+4h&=360 &&\text{(refining plant)}
\end{aligned}$$

   (b) Solve the equations to show that we can make 40 tons of l and 10 tons of h.

Solution. Using an augmented matrix form, we have

$$\begin{matrix}\text{a}\\ \text{b}\end{matrix}\begin{pmatrix}10 & 8 & 480\\ 8 & 4 & 360\end{pmatrix}\sim\begin{matrix}\text{a}-2\text{b}\\ -4\text{a}+5\text{b}\end{matrix}\begin{pmatrix}-6 & 0 & -240\\ 0 & -12 & -120\end{pmatrix},$$

and so we can make 40 tons of l and 10 tons of h.

6. You are given three alloys of the following composition:
   Alloy A: 5 parts (by weight) gold, 2 silver, 1 lead
   Alloy B: 2 parts gold, 5 silver, 1 lead
   Alloy C: 3 parts gold, 1 silver, 4 lead

   (a) How much of each metal is in one ounce of Alloy A?

Solution. Adding up the number of parts in alloy A, we have eight. Therefore, each ounce of alloy A has 5/8 ounce gold, 1/4 ounce silver, and 1/8 ounce lead.

Suppose we want to make 27 ounces of a new alloy containing equal quantities (by weight) of gold, silver, and lead.

(b) Write the system of equations needed to solve for the amounts of A, B, and C necessary to make this new alloy.

Solution. Using the above reasoning for alloys B and C, we have

$$\frac{5A}{8}+\frac{B}{4}+\frac{3C}{8}=9 \qquad \text{(gold equation)}$$
$$\frac{A}{4}+\frac{5B}{8}+\frac{C}{8}=9 \qquad \text{(silver equation)}$$
$$\frac{A}{8}+\frac{B}{8}+\frac{C}{2}=9 \qquad \text{(lead equation)}$$

(c) Solve the equations to show that we need 15 ounces of C, 11 ounces of B, and 1 ounce of A.

Solution. Using an augmented matrix form, we have

$$\begin{matrix}\mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\end{matrix}\begin{pmatrix}5/8 & 1/4 & 3/8 & 9\\ 1/4 & 5/8 & 1/8 & 9\\ 1/8 & 1/8 & 1/2 & 9\end{pmatrix} \sim \begin{matrix}\mathrm{d}=8\mathrm{a}\\ \mathrm{e}=\mathrm{b}-2\mathrm{c}\\ \mathrm{f}=\mathrm{a}-5\mathrm{c}\end{matrix}\begin{pmatrix}5 & 2 & 3 & 72\\ 0 & 3/8 & -7/8 & -9\\ 0 & -3/8 & -17/8 & -36\end{pmatrix},$$

$$\sim \begin{matrix}\mathrm{d}\\ 8\mathrm{e}\\ \mathrm{e}+\mathrm{f}\end{matrix}\begin{pmatrix}5 & 2 & 3 & 72\\ 0 & 3 & -7 & -72\\ 0 & 0 & -3 & -45\end{pmatrix},$$

and so we need 15 ounces of C, 11 ounces of B, and 1 ounce of A.

7. Suppose that the following facts are true:
   (1) Of the number of people x_{s} who start a year living on the East Coast, 80% stay in and 20% move out during the course of that year.
   (2) Of the number of people y_{s} who start a year living off the East Coast, 90% stay out and 10% move in during the course of that year.
   (3) The number of births and deaths cancel one another out (so you don't have to worry about them).

   Let x_{e} and y_{e} be the number of people living on and off the East Coast, respectively, at the *end* of the year.

   (a) Write the system of equations needed to solve for x_{e} and y_{e} as functions of x_{s} and y_{s}.

Solution.

$$\begin{aligned}x_{\mathrm{e}}&=0.8x_{\mathrm{s}}+0.1y_{\mathrm{s}}\\ y_{\mathrm{e}}&=0.2x_{\mathrm{s}}+0.9y_{\mathrm{s}}\end{aligned} \qquad \text{(D)}$$

(b) If $y_{\mathrm{e}}=155$ million and $x_{\mathrm{e}}=95$ million, find x_{s} and y_{s}.

Solution. Using an augmented matrix form, we have

$$\begin{matrix}\mathrm{a}\\ \mathrm{b}\end{matrix}\begin{pmatrix}0.8 & 0.1 & 95\\ 0.2 & 0.9 & 155\end{pmatrix} \sim \begin{matrix}\mathrm{c}=10\mathrm{a}\\ \mathrm{d}=\mathrm{a}-4\mathrm{b}\end{matrix}\begin{pmatrix}8 & 1 & 950\\ 0 & -3.5 & -525\end{pmatrix} \sim \begin{matrix}\mathrm{c}-\mathrm{e}\\ \mathrm{e}=-\mathrm{d}/3.5\end{matrix}\begin{pmatrix}8 & 0 & 800\\ 0 & 1 & 150\end{pmatrix},$$

so we have $y_{\mathrm{s}} = 150$ million, $x_{\mathrm{s}} = 100$ million.

(c) If $x_{\mathrm{s}} = x_{\mathrm{e}}$ and $y_{\mathrm{s}} = y_{\mathrm{e}}$, show that the ratio $y_{\mathrm{e}}/x_{\mathrm{e}} = 2$.

Solution. Rewriting (D) in this case, we have

$$\begin{aligned}-0.2x_{\mathrm{e}} + 0.1y_{\mathrm{e}} &= 0\\ 0.2x_{\mathrm{e}} - 0.1y_{\mathrm{e}} &= 0,\end{aligned}$$

which is a consistent system and has the solution $y_{\mathrm{e}}/x_{\mathrm{e}} = 2$.

8. Consider the system

$$\begin{aligned}x - \ \ 3y + \ \ \ z &= 1\\ 2x + \ \ 2y - \ \ 3z &= 2\\ -5x - 17y + 15z &= b,\end{aligned} \tag{5.4}$$

where b is a constant.

(a) Write the system in augmented matrix form, and row reduce it to the following form:

$$\begin{pmatrix}1 & 0 & * & *\\ 0 & 1 & * & *\\ 0 & 0 & * & *\end{pmatrix},$$

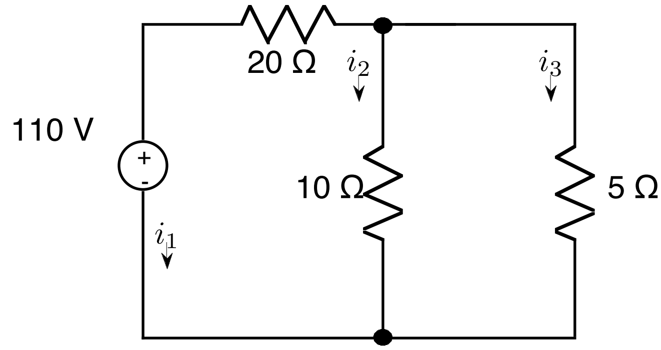
where the * are unknown entries that you must find.

Solution. Using an augmented matrix form, we have

$$\begin{matrix}\mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\end{matrix}\begin{pmatrix}1 & -3 & 1 & 1\\ 2 & 2 & -3 & 2\\ -5 & -17 & 15 & b\end{pmatrix} \sim \begin{matrix}\mathrm{a}\\ \mathrm{d}=\mathrm{b}-2\mathrm{a}\\ \mathrm{e}=\mathrm{c}+5\mathrm{a}\end{matrix}\begin{pmatrix}1 & -3 & 1 & 1\\ 0 & 8 & -5 & 0\\ 0 & -32 & 20 & b+5\end{pmatrix}$$

$$\sim \begin{matrix}\mathrm{a}+3\mathrm{f}\\ \mathrm{f}=\mathrm{d}/8\\ \mathrm{e}+4\mathrm{d}\end{matrix}\begin{pmatrix}1 & 0 & -7/8 & 1\\ 0 & 1 & -5/8 & 0\\ 0 & 0 & 0 & b+5\end{pmatrix}.$$

(b) Find all values of b for which (5.4) has a solution, and write the solution in that case.



Solution. The last row above indicates that we have a solution if and only if $b = -5$. In that case, we may let z be the free variable to write our solution as

$$(x, y, z) = \left(1 + \frac{7z}{8}, \frac{5z}{8}, z\right)$$

9. Solve for the currents in the network above.

Solution. Balancing the currents at the top node, we have

$$\begin{aligned} i_1 + i_2 + i_3 &= 0 && \text{(current balance)} \\ -20i_1 + 10i_2 &= 110 && \text{(left voltage loop)} \\ -10i_2 + 5i_3 &= 0 && \text{(right voltage loop)} \end{aligned}$$

Rewriting in augmented matrix form, we have

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ -20 & 10 & 0 & 110 \\ 0 & -10 & 5 & 0 \end{pmatrix} \sim \begin{matrix} \text{a} \\ \text{d} = \text{b} + 20\text{a} \\ \text{e} = 3\text{c} + \text{d} \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 30 & 20 & 110 \\ 0 & 0 & 35 & 110 \end{pmatrix}$$

$$\sim \begin{matrix} \text{a} \\ \text{d}/10 \\ \text{e}/5 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 11 \\ 0 & 0 & 7 & 22 \end{pmatrix},$$

and so we have $i_3 = 22/7$, $i_2 = 11/7$, $i_1 = -33/7$.

10. Let $A \in \mathcal{R}^{m \times n}$, $\mathbf{b} \in \mathcal{R}^m$, $\mathbf{x} \in \mathcal{R}^n$. Moreover, let $A\mathbf{x} = \mathbf{b}$, and $\mathbf{z} = 3\mathbf{b}$. Does the equation $A\mathbf{y} = \mathbf{z}$ have a solution? If so, compute it. If not, explain why not.

Solution. By property V2, $A(3\mathbf{x}) = 3(A\mathbf{x}) = 3\mathbf{b} = \mathbf{z}$. So $\mathbf{y} = 3\mathbf{x}$.