Homework Set 4 Solutions

1. A mass weighing 96 pounds stretches a spring 8 feet.

   (a) Determine the amplitude and period of motion if the mass is initially released from a point 2 feet below the equilibrium position with a downward velocity of 1 ft/s.

Solution. By notes in class, we have that the spring constant is given by

$$k = \frac{\text{weight}}{\text{stretch}} = \frac{96 \text{ lb}}{4 \text{ ft}} = 12 \, \frac{\text{lb}}{\text{ft}}.$$

Moreover, we have that the mass is given by

$$M = \frac{\text{weight}}{32} = \frac{96}{32} = 3.$$

Then the equation to solve becomes

$$3\ddot{x} + 12x = 0. \tag{A.1}$$

Moreover, since "down" corresponds to extension, or positive x, we have

$$x(0) = 2, \quad \dot{x}(0) = 1, \tag{A.2}$$

where feet is the underlying unit of distance. Substituting $x = e^{\lambda t}$ into (A), we have

$$\begin{aligned}
3(\lambda^2 + 4) = 0 \qquad &\Longrightarrow \qquad \lambda_\pm = \pm 2i \\
x(t) &= c_1 \cos 2t + c_2 \sin 2t \\
x(0) &= c_1 = 2 \\
\dot{x}(0) &= 2c_2 = 1 \qquad \Longrightarrow \qquad c_2 = \frac{1}{2} \\
x(t) &= 2\cos 2t + \frac{\sin 2t}{2} = \sqrt{4 + \frac{1}{4}} \cos(2t - \phi), \qquad \tan\phi = \frac{c_2}{c_1} = \frac{1/2}{2}.
\end{aligned}$$

Then keeping in mind that $c_1 > 0$, we have

$$x(t) = \frac{\sqrt{17}}{2} \cos(2t - \phi), \qquad \phi = \tan^{-1} \frac{1}{4} \approx 0.2445$$

Hence the amplitude is $\sqrt{17}/2$ feet. In one period T, $2T = 2\pi$, so $T = \pi$.

(b) How many complete cycles will the mass have made at the end of 4π seconds?

Solution. The number of cycles is given by

$$\frac{4\pi}{T} = \frac{4\pi}{\pi} = 4.$$

2. After a mass weighing 25 pounds is attached to a 8-foot spring, the spring measures 13 feet. This mass is removed and replaced with another mass that weighs 16 pounds. The entire system is placed in a medium that offers a damping force numerically equal to the instantaneous velocity.

   (a) Find the equation of motion if the mass is initially released from a point 1 foot below the equilibrium position with an upward velocity of -4 ft/s.

Solution. By notes in class, we have that the spring constant is given by

$$k = \frac{\text{weight}}{\text{stretch}} = \frac{25\text{ lb}}{(13-8)\text{ ft}} = \frac{25}{5}\frac{\text{lb}}{\text{ft}} = 5\,\frac{\text{lb}}{\text{ft}}.$$

Moreover, we have that the new mass is given by

$$M = \frac{\text{weight}}{32} = \frac{16}{32} = \frac{1}{2}.$$

Then the equation to solve becomes

$$\frac{1}{2}\ddot{x} + \dot{x} + 5x = 0, \tag{B.1}$$

where we have used the fact that the size of the damping force is equal to the speed. Moreover, since "down" corresponds to extension, or positive x, we have

$$x(0) = 1, \quad \dot{x}(0) = -4, \tag{B.2}$$

where feet is the underlying unit of distance. Substituting $x = e^{\lambda t}$ into (B), we have

$$\frac{\lambda^2}{2} + \lambda + 5 = 0 \qquad \Longrightarrow \qquad \lambda_{\pm} = \frac{-1 \pm \sqrt{1 - 4(5)/2}}{2/2} = -1 \pm 3i$$

$$\begin{aligned}
x(t) &= e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right)\\
x(0) &= c_1 = 1\\
\dot{x}(0) &= 3c_2 - c_1 = -4 \qquad \Longrightarrow \qquad c_2 = -1\\
x(t) &= e^{-t}(\cos 3t - \sin 3t).
\end{aligned}$$

(b) Write your solution in the amplitude-phase form given in class.

Solution.

$$x(t) = e^{-t}\sqrt{1+1}\cos(3t - \phi), \qquad \tan\phi = \frac{c_2}{c_1} = -1.$$

Then keeping in mind that $c_1 > 0$, $c_2 < 0$, we have

$$x(t) = \sqrt{2}e^{-t}\cos(3t - \phi), \qquad \phi = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

3. A mass weighing 36 pounds stretches a spring 2 feet. The mass is then attached to a dashpot that damps the motion with a force equal to β times the velocity, where β is a positive constant.

   (a) Write the equation of motion for the displacement $x(t)$.

Solution. By notes in class, we have that the spring constant is given by

$$k = \frac{\text{weight}}{\text{stretch}} = \frac{36 \text{ lb}}{2 \text{ ft}} = 18 \,\frac{\text{lb}}{\text{ft}}.$$

Moreover, we have that the mass is given by

$$M = \frac{\text{weight}}{32} = \frac{36}{32} = \frac{9}{8}.$$

Then the equation to examine becomes

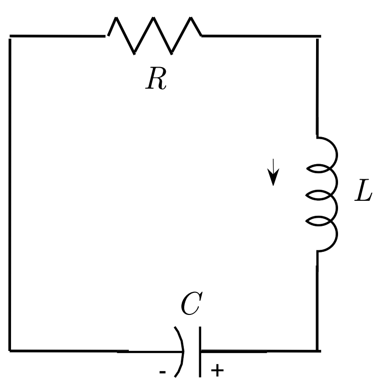
$$\frac{9}{8}\ddot{x} + \beta\dot{x} + 18x = 0. \tag{C}$$

   (b) Find the values of β such that the motion is overdamped, underdamped, and critically damped.

Solution. Substituting $x = e^{\lambda t}$ into (C), we obtain

$$\frac{9\lambda^2}{8} + \beta\lambda + 18 = 0 \qquad \Longrightarrow \qquad \lambda_\pm = \frac{-\beta \pm \sqrt{\beta^2 - 18(9/2)}}{2(9/8)} = \frac{-\beta \pm \sqrt{\beta^2 - 9^2}}{9/4}.$$

Therefore, our solutions are of the form $x = c_+e^{\lambda_+ t} + c_-e^{\lambda_- t}$. The solutions are overdamped if the square root is real, so when $\beta > 9$. The solutions are underdamped if the square root is imaginary, so when $\beta < 9$. The solutions are critically damped when $\beta = 9$.



4. Consider the *series RLC* circuit shown in the figure above. There is an initial voltage on the capacitor of V_0.

   (a) Use the fact that the sum of the voltages around the loop must be zero to obtain the ODE
$$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0. \tag{4.1a}$$

Solution. From Homework Set 1, we know that the voltage through the inductor, resistor, and capacitor are described by $V_L = L\dot{I}$, $V_R = IR$, $\dot{V}_c = I/C$, respectively. Integrating the last equation, we have that

$$V_c = \int_0^t \frac{I}{C}\,dt + V_0.$$

Since the voltage around the loop must be zero, we have

$$L\dot{I} + IR + \int_0^t \frac{I}{C}\,dt + V_0 = 0. \tag{D}$$

Taking the derivative of (D) with respect to t yields (4.1a).

Initially, the current is I_0.

   (b) Show that the resulting initial conditions are

$$I(0) = I_0, \qquad \dot{I}(0) = -\frac{V_0 + RI_0}{L}. \tag{4.1b}$$

Solution. The first equation follows trivially from the problem statement. Substituting this result and the initial condition for the voltage into (D) evaluated at $t = 0$, we have

$$L\dot{I}(0) + I(0)R + V_0 = 0,$$

where we have used the fact that the integral vanishes. Then using the value for $I(0)$, we have
$$\dot{I}(0) = -\frac{V_0 + RI_0}{L}.$$

   (c) Under what conditions will the circuit be underdamped? overdamped? critically damped?

Solution. Substituting $I = e^{\lambda t}$ into (4.1a), we obtain

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \qquad \Longrightarrow \qquad \lambda_\pm = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}. \tag{E}$$

Therefore, our solutions are of the form $I = c_+e^{\lambda_+ t} + c_-e^{\lambda_- t}$. The solutions are overdamped if the square root is real, so when $R^2 > 4L/C$. The solutions are underdamped if the square root is imaginary, so when $R^2 < 4L/C$. The solutions are critically damped when $R^2 = 4L/C$.

(d) Solve the system when $L = 1$, $C = 1/5$, $I_0 = 2$, $R = V_0 = 4$.

Solution. Substituting our parameters into (E), we obtain

$$\lambda_{\pm} = -\frac{4}{2} \pm \frac{1}{2}\sqrt{16 - 20} \qquad \Longrightarrow \qquad I(t) = e^{-2t}(c_1 \cos t + c_2 \sin t).$$

Solving the initial conditions, we obtain

$$I(0) = c_1 = I_0 = 2$$

$$\begin{aligned} e^{-2t}(-2\sin t + c_2 \cos t) - 2e^{-2t}(2\cos t + c_2 \sin t)\big|_{t=0} &= -\frac{V_0 + RI_0}{L} = -\frac{4+8}{1} \\ c_2 - 4 &= -12 \\ c_2 &= -8 \\ I(t) &= e^{-2t}(2\cos t - 8\sin t). \end{aligned}$$

5. Consider two species in a closed environment: a predator (population f) and its prey (population h). An early model for the evolution of the populations is, after suitable nondimensionalization,

$$\dot{h} = h(1 - f), \tag{4.2a}$$
$$\dot{f} = \alpha^2 f(h - 1). \tag{4.2b}$$

One *fixed point* of this equation (where $\dot{h} = \dot{f} = 0$) is $(h, f) = (0, 0)$. (Clearly if there are no specimens to begin with, the size of the populations will not change.)

(a) Find the other, more realistic fixed point (h_*, f_*).

Solution. $\dot{h} = 0$ also when $f = 1$, and $\dot{f} = 0$ also when $h = 1$. Therefore $(h_*, f_*) = (1, 1)$.

To examine what happens in the neighborhood of the populations near the fixed point in (a), we let

$$\epsilon x(t) = h(t) - h_*, \quad \epsilon y(t) = f(t) - f_*; \qquad 0 < \epsilon \ll 1. \tag{4.3}$$

(b) Show that if we substitute (4.3) into (4.2) and take the limit that $\epsilon \to 0$, the resulting system for x and y is

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= \alpha^2 x. \end{aligned} \tag{4.4}$$

Solution. Rewriting (4.3) with our values of $f_* = h_* = 1$, we have

$$h(t) = 1 + \epsilon x(t), \qquad f(t) = 1 + \epsilon y(t).$$

Substituting these expressions into (4.2), we obtain

$$\begin{aligned} \epsilon\dot{x} &= (1+\epsilon x)[1-(1+\epsilon y)], & \epsilon\dot{y} &= \alpha^2(1+\epsilon y)[(1+\epsilon x)-1] \\ \dot{x} &= -y(1+\epsilon x), & \dot{y} &= \alpha^2 x(1+\epsilon y) \\ \dot{x} &= -y, & \dot{y} &= \alpha^2 x, \end{aligned}$$

where in the last lines, we have taken the limit that $\epsilon \to 0$.

(c) Reduce the system (4.4) to a single second-order ODE for $x(t)$.

Solution. Taking the derivative of the top equation, we have that $\ddot{x} = -\dot{y}$, and so the second equation becomes

$$-\ddot{x} = \alpha^2 x. \tag{F}$$

(d) Show that the solution to the system (4.4) is given by

$$\begin{aligned} x(t) &= c_1 \cos\alpha t + c_2 \sin\alpha t, \\ y(t) &= \alpha(c_1 \sin\alpha t - c_2 \cos\alpha t). \end{aligned}$$

Solution. Rewriting (F), we have

$$\ddot{x} + \alpha^2 x = 0 \qquad \Longrightarrow \qquad x(t) = c_1 \cos\alpha t + c_2 \sin\alpha t.$$

Subsituting this expression into the first equation in (4.4), we have

$$y = -\alpha(-c_1 \sin\alpha t + c_2 \cos\alpha t),$$

from which the desired result immediately follows. Note that we should *not* substitute into the second equation, as this would give us a spurious additional constant of integration.

6. Find the general solution to the differential equation

$$\ddot{y} + y = \sin\omega t.$$

Be sure to account for all ω.

Solution. Using the method of undetermined coefficients, we try to find a particular solution of the form

$$y_p = c_{\text{c}} \cos\omega t + c_{\text{s}} \sin\omega t.$$

Substituting in this form, we obtain

$$\begin{aligned} -c_{\text{c}}\omega^2 \cos\omega t - c_{\text{s}}\omega^2 \sin\omega t + c_{\text{c}} \cos\omega t + c_{\text{s}} \sin\omega t &= \sin\omega t \\ c_{\text{c}}(1-\omega^2)\cos\omega t + c_{\text{s}}(1-\omega^2)\sin\omega t &= \sin\omega t \end{aligned}$$

$$c_{\text{c}} = 0, \qquad c_{\text{s}} = \frac{1}{1-\omega^2}, \qquad \omega \neq \pm 1.$$

For the case where $\omega = \pm 1$, we try

$$y_p = s_1 t \sin \omega t + c_1 t \cos \omega t.$$

Substituting in this form, we obtain

$$\begin{aligned} 2s_1\omega \cos \omega t - \omega^2 s_1 t \sin \omega t - 2c_1\omega \sin \omega t - \omega^2 c_1 t \cos \omega t + \omega^2[s_1 t \sin \omega t + c_1 t \cos \omega t] &= \sin \omega t \\ 2s_1\omega \cos \omega t - 2c_1\omega \sin \omega t &= \sin \omega t \end{aligned}$$

$$c_1 = -\frac{1}{2\omega}, \qquad s_1 = 0.$$

The homogeneous solution is trivially given by

$$y_h = A \cos t + B \sin t.$$

Therefore, the general solution is given by

$$y(t) = y_p(t) + A \cos t + B \sin t, \qquad y_p(t) = \begin{cases} \dfrac{\sin \omega t}{1-\omega^2}, & \omega \neq \pm 1, \\ -\dfrac{t \cos \omega t}{2\omega}, & \omega = \pm 1. \end{cases}$$

7. Find the solution to the system

$$\ddot{y} + 16y = 32t^2, \qquad y(0) = 0, \quad \dot{y}(0) = 12.$$

Solution. Using the method of undetermined coefficients, we try to find a particular solution of the form

$$y_{\text{p}} = a_2 t^2 + a_1 t + a_0.$$

Substituting in this form, we obtain

$$2a_2 + 16(a_2 t^2 + a_1 t + a_0) = 32t^2.$$

Matching powers of t, we have that $a_1 = 0$. Matching powers of t^2, we have $16a_2 = 32$, so $a_2 = 2$. Using these results to match the constant terms, we have

$$2(2) + 16a_0 = 0 \qquad \Longrightarrow \qquad a_0 = -\frac{1}{4}.$$

Using our previous homework with $\omega = 4$, we have that the homogeneous solution is given by

$$y_{\text{h}} = c_1 \cos 4t + c_2 \sin 4t.$$

Therefore, the general solution is given by

$$y(t) = 2t^2 - \frac{1}{4} + c_1 \cos 4t + c_2 \sin 4t.$$

Substituting this result into the boundary conditions, we have

$$y(0) = -\frac{1}{4} + c_1 = 0 \qquad \dot{y}(0) = 4c_2 = 12$$
$$c_1 = \frac{1}{4}, \qquad c_2 = 3.$$

$$y(t) = 2t^2 + \frac{\cos 4t - 1}{4} + 3\sin 4t.$$

8. Find the general solution to the differential equation

$$\ddot{y} + 3\dot{y} + 2y = 12e^{2t} + 18e^{t}$$

using the method of undetermined coefficients.

Solution. Using the method of undetermined coefficients, we try to find a particular solution of the form

$$y_p = c_2 e^{2t} + c_1 e^{t}.$$

Substituting in this form, we obtain

$$(4c_2e^{2t} + c_1e^t) + 3(2c_2e^{2t} + c_1e^t) + 2(4c_2e^{2t} + c_1e^t) = 12e^{2t} + 18e^t$$
$$12c_2e^{2t} + 6c_1e^t = 12e^{2t} + 18e^t$$
$$c_1 = 3, \quad c_2 = 1.$$

By substituting $y = e^{\lambda t}$, we can obtain the homogeneous solution, where λ solves

$$\lambda^2 + 3\lambda + 2 = 0 \qquad \Longrightarrow \qquad \lambda = -2, -1$$

Therefore, the general solution is given by

$$y(t) = e^{2t} + 3e^t + Ae^{-2t} + Be^{-t}.$$

9. Solve

$$\ddot{y} - 9y = 6e^{3t}$$

by the method of undetermined coefficients.

Solution. Substituting $y = e^{\lambda t}$ to find the homogeneous solution, we have

$$\lambda^2 - 9 = 0 \qquad \Longrightarrow \qquad \lambda = \pm 3 \qquad \Longrightarrow \qquad y_{\mathrm{h}}(t) = c_1e^{3t} + c_2e^{-3t}.$$

Therefore, we see that the right-hand side is a multiple of a homogeneous solution, so we need to try a particular solution of the form

$$y_{\mathrm{p}}(t) = Ate^{3t} \qquad \Longrightarrow \qquad \dot{y}_{\mathrm{p}}(t) = A(3t+1)e^{3t} \qquad \Longrightarrow \qquad \ddot{y}_{\mathrm{p}}(t) = A[3(3t+1)+3]e^{3t}.$$

Substituting in this form, we obtain

$$\begin{aligned}A(9t+6)e^{3t} - 9Ate^{3t} &= 6e^{3t}\\ 6A &= 6\\ y(x) &= c_1e^{3t} + c_2e^{-3t} + te^{3t}.\end{aligned}$$

10. Consider the equation

$$\ddot{y} + 3\dot{y} = 10e^{-2t}\sin t, \qquad y(0) = 3, \qquad \dot{y}(0) = -2.$$

(a) Find the solution.

Solution. We try a particular solution of the form

$$y_{\mathrm{p}} = c_{\mathrm{c}}e^{-2t}\cos t + c_{\mathrm{s}}e^{-2t}\sin t.$$

Substituting in this form, we obtain

$$\begin{aligned}(4c_{\mathrm{c}}e^{-2t}\cos t + 4c_{\mathrm{s}}e^{-2t}\sin t - 4c_{\mathrm{s}}e^{-2t}\cos t + 4c_{\mathrm{c}}e^{-2t}\sin t - c_{\mathrm{c}}e^{-2t}\cos t - c_{\mathrm{s}}e^{-2t}\sin t)&\\ + 3(-2c_{\mathrm{c}}e^{-2t}\cos t - 2c_{\mathrm{s}}e^{-2t}\sin t + c_{\mathrm{s}}e^{-2t}\cos t - c_{\mathrm{c}}e^{-2t}\sin t) &= 10e^{-2t}\sin t\\ (c_{\mathrm{s}} - 3c_{\mathrm{c}})e^{-2t}\cos t + (-c_{\mathrm{c}} - 3c_{\mathrm{s}})e^{-2t}\sin t &= 10e^{-2t}\sin t\\ -c_{\mathrm{s}} - 3c_{\mathrm{c}} &= 0\\ c_{\mathrm{c}} - 3c_{\mathrm{s}} &= 10.\end{aligned}$$

Solving the last two equations together, we have $c_{\mathrm{s}} = -3$, $c_{\mathrm{c}} = 1$. By substituting $y = e^{\lambda t}$, we can obtain the homogeneous solution, where λ solves

$$\lambda^2 + 3\lambda = \lambda(\lambda+3) = 0.$$

Thus, we have

$$y(t) = e^{-2t}\cos t - 3e^{-2t}\sin t + Ae^{-3t} + B.$$

Solving the initial data, we obtain

$$\begin{aligned}y(0) &= 1 + A + B = 3\\ \dot{y}(0) &= -2 - 3 - 3A = -2.\end{aligned}$$

Therefore, we have that $A = -1$, $B = 3$, so the solution is

$$y(t) = e^{-2t}\cos t - 3e^{-2t}\sin t - e^{-3t} + 3. \tag{G}$$

(b) Show that $y(\pi) \approx 3$.

Solution. Substituting $y = \pi$ into (G), we obtain

$$y(\pi) = -e^{-2\pi} - e^{-3\pi} + 3 \approx 3,$$

since the first two terms are exponentially small.