Homework Set 3 Solutions

1. For the equation
$$\ddot{y} + 3\dot{y} - 10y = 0,$$
find the fundamental set $\{y_1(t), y_2(t)\}$ where
$$y_1(0) = 1, \quad \dot{y}_1(0) = 0; \qquad y_2(0) = 0, \quad \dot{y}_2(0) = 1.$$

Solution. Substituting $y = e^{\lambda t}$, we obtain
$$\begin{gathered}\lambda^2 + 3\lambda - 10 = 0 \\ (\lambda + 5)(\lambda - 2) = 0,\end{gathered}$$
so solutions are of the form $y = c_1 e^{-5t} + c_2 e^{2t}$. Therefore, for y_1 we must solve
$$\begin{aligned} c_1 + c_2 &= 1 \\ -5c_1 + 2c_2 &= 0. \end{aligned} \qquad \Longrightarrow \qquad c_1 = \frac{2}{7}, \quad c_2 = \frac{5}{7} \qquad \Longrightarrow \qquad y_1(t) = \frac{2e^{-5t} + 5e^{2t}}{7},$$
and for y_2 we must solve
$$\begin{aligned} c_1 + c_2 &= 0 \\ -5c_1 + 2c_2 &= 1. \end{aligned} \qquad \Longrightarrow \qquad c_1 = -\frac{1}{7}, \quad c_2 = \frac{1}{7} \qquad \Longrightarrow \qquad y_2(t) = \frac{e^{2t} - e^{-5t}}{7}.$$

2. Consider the equation
$$(\sin t)\ddot{y} + t\dot{y} + \frac{3y}{t^2 + 2t - 3} = 0.$$
Find all intervals where this equation is guaranteed to have a unique solution.

Solution. Rewriting the equation in standard form, we have
$$\ddot{y} + \frac{t}{\sin t}\dot{y} + \frac{3y}{(t+3)(t-1)\sin t} = 0.$$
The coefficient of y is undefined whenever $t = 1$, $t = -3$, and $t = n\pi$, where n is an integer. Therefore, the equation has a unique solution in any interval not containing those points.

3. Consider the ODE
$$y^{(3)} - 7\dot{y} + 6y = 0. \tag{3.1}$$

(a) By direct substitution, show that three solutions of (3.1) are given by $\{e^t, e^{2t}, e^{-3t}\}$.

Solution. Plugging in each of the three solutions listed, we obtain

$$\begin{aligned} \frac{d^3(e^t)}{dt^3} - 7\frac{d(e^t)}{dt} + 6e^t &= (1-7+6)e^t = 0, \\ \frac{d^3(e^{2t})}{dt^3} - 7\frac{d(e^{2t})}{dt} + 6e^{2t} &= (8-14+6)e^{2t} = 0, \\ \frac{d^3(e^{-3t})}{dt^3} - 7\frac{d(e^{-3t})}{dt} + 6e^{-3t} &= (27-21+6)e^{-3t} = 0, \end{aligned}$$

as required.

(b) Show that the Wronskian of these three solutions is constant.

Solution. Remember that we calculate the Wronskian by putting the functions in the first row, the derivatives in the second row, and the second derivatives in the third row. Here is one solution:

$$\begin{aligned} W\{e^t, e^{2t}, e^{-3t}\} &= \begin{vmatrix} e^t & e^{2t} & e^{-3t} \\ e^t & 2e^{2t} & -3e^{-3t} \\ e^t & 2^2e^{2t} & (-3)^2e^{-3t} \end{vmatrix} \\ &= e^t \begin{vmatrix} 2e^{2t} & -3e^{-3t} \\ 4e^{2t} & 9e^{-3t} \end{vmatrix} - e^t \begin{vmatrix} e^{2t} & e^{-3t} \\ 4e^{2t} & 9e^{-3t} \end{vmatrix} + e^t \begin{vmatrix} e^{2t} & e^{-3t} \\ 2e^{2t} & -3e^{-3t} \end{vmatrix} \\ &= e^t(18e^{-t} + 12e^{-t}) - (9e^{-t} - 4e^{-t}) + e^t(-3e^{-t} - 2e^{-t}) = 20. \end{aligned}$$

4. Consider the equation

$$3\ddot{y} + 7\dot{y} + 2y = 0.$$

(a) Find the general solution.

Solution. Substituting $e^{\lambda t}$ into the above, we have

$$3\lambda^2 + 7\lambda + 2 = (3\lambda+1)(\lambda+2) = 0 \qquad \Longrightarrow \qquad \lambda_1 = -\frac{1}{3}, \quad \lambda_2 = -2.$$

Hence we have that

$$y(t) = c_1e^{-t/3} + c_2e^{-2t}.$$

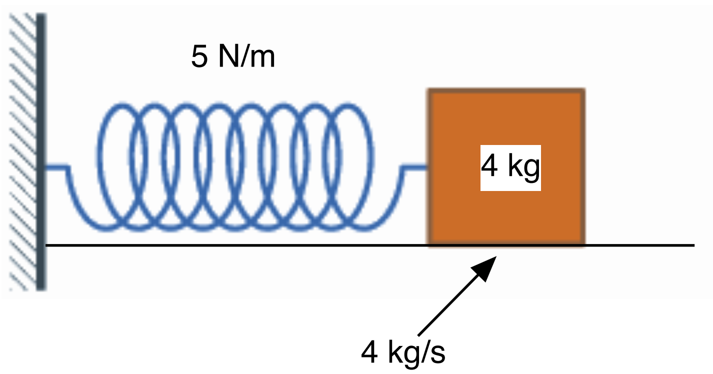
(b) Describe the behavior of the solution as $t \to \infty$. Is the solution overdamped, underdamped, critically damped, or none of those?

Solution. Both roots are negative and real, so the solution goes to 0 as $t \to \infty$. Hence the solution is overdamped.

5. A spring with stiffness $k = 5$ N/m is damped with damping constant $b = 4$ kg/s and is attached to a weight with mass $M = 4$ kg (see figure).

(a) Show that the resulting equation for the displacement x is

$$4\ddot{x} + 4\dot{x} + 5x = 0. \tag{3.2}$$



Solution. From notes in class, we have that the general equation for a damped spring

$$M\ddot{x} + b\dot{x} + kx = 0.$$

Therefore, substituting in the parameter values given, we obtain

$$4\ddot{x} + 4\dot{x} + 5x = 0,$$

as required.

The initial displacement is 2 m, and the initial velocity is −2 m/s.

(b) Solve (3.2) subject to these conditions.

Solution. Substituting $x = e^{\lambda t}$, we obtain

$$4\lambda^2 + 4\lambda + 5 = 0 \qquad \Longrightarrow \qquad \lambda_\pm = -\frac{1}{2} \pm i,$$

so solutions are of the form $x = e^{-t/2}(c_1 \cos t + c_2 \sin t)$. Solving the first initial condition, we immediately have that

$$x(0) = c_1 = 2.$$

Solving the second initial condition, we have

$$\begin{aligned} e^{-t/2}(-2\sin t + c_2 \cos t) - e^{-t/2}[\cos t + (c_2/2)\sin t]\Big|_{t=0} &= -2 \\ c_2 - 1 &= -2 \\ c_2 &= -1 \\ x(t) &= e^{-t/2}(2\cos t - \sin t). \end{aligned}$$

(c) Equipment in the lab can measure the displacement down to a level of 1 mm. Estimate the time t_* after which the displacement will *always* remain below the threshold level.

Solution. Transforming into amplitude-phase form, we have

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{2^2 + (-1)^2} = \sqrt{5}, \qquad \phi = \tan^{-1}(-1/2),$$

where we don't have to add the extra π term since $c_1 > 0$. Therefore, we have

$$x(t) = e^{-t/2}\sqrt{5}\cos(t - \phi),$$

where we have to multiply by the decay term $e^{-t/2}$. Hence by equating the amplitude to 10^{-3} m, we obtain the desired estimate:

$$\begin{aligned} e^{-t/2}\sqrt{5} &= 0.001 \\ -\frac{t}{2} &= \log\left(\frac{10^{-3}}{\sqrt{5}}\right) \\ t &= 15.4 \text{ s.} \end{aligned}$$

6. Prove the following statements:

   (a)

$$y(t) = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t) \qquad \Longrightarrow \qquad \dot{y}(0) = \alpha c_1 + \beta c_2.$$

Solution.

$$\begin{aligned} \dot{y} &= \alpha e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t) + e^{\alpha t}(-\beta c_1 \sin\beta t + \beta c_2 \cos\beta t) \\ \dot{y}(0) &= \alpha(c_1 \cdot 1 + c_2 \cdot 0) + \beta(-c_1 \cdot 0 + c_2 \cdot 1) = \alpha c_1 + \beta c_2. \end{aligned}$$

   (b)

$$y(t) = e^{\alpha t}(c_1 t + c_2) \qquad \Longrightarrow \qquad \dot{y}(0) = c_1 + \alpha c_2.$$

Solution.

$$\begin{aligned} \dot{y} &= \alpha e^{\alpha t}(c_1 t + c_2) + e^{\alpha t}(c_1) \\ \dot{y}(0) &= c_1 + \alpha c_2. \end{aligned}$$

7. Consider the equation

$$\ddot{x} + 2\dot{x} + 2x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = \alpha \geq 0.$$

   (a) Construct the solution $x(t)$.

Solution. Substituting $x = e^{\lambda t}$, we obtain

$$\lambda^2 + 2\lambda + 2 = 0 \qquad \Longrightarrow \qquad \lambda_\pm = -1 \pm i,$$

so solutions are of the form $x = e^{-t}(c_1 \cos t + c_2 \sin t)$. Solving the first initial condition, we immediately have that

$$x(0) = c_1 = 1.$$

Solving the second initial condition, we have

$$\begin{aligned} e^{-t}(-\sin t + c_2\cos t) - e^{-t}(\cos t + c_2 \sin t)\big|_{t=0} &= \alpha \\ c_2 - 1 &= \alpha \\ x(t) &= e^{-t}[\cos t + (\alpha+1)\sin t]. \end{aligned}$$

(b) Show that $x(t_*) = 0$ whenever

$$\tan t_* = -\frac{1}{\alpha+1}. \tag{3.3}$$

Solution.

$$\begin{aligned} x(t_*) = e^{-t_*}[\cos t_* + (\alpha+1)\sin t_*] &= 0 \\ \cos t_* &= -(\alpha+1)\sin t_* \\ \tan t_* &= -\frac{1}{\alpha+1}. \end{aligned}$$

(c) By considering the limits $\alpha \to 0$ and $\alpha \to \infty$, construct upper and lower bounds on the smallest positive t_*.

Solution. As $\alpha \to 0$, the right-hand side of (3.3) approaches -1. The first positive t_* for which $\tan t_* = -1$ is $3\pi/4$. As $\alpha \to \infty$, the right-hand side of (3.3) approaches 0. The first positive t_* for which $\tan t_* = 0$ is π. Therefore, we have that

$$\frac{3\pi}{4} \le t_* < \pi,$$

where we include the equality only for the attainable limit $\alpha = 0$.

8. If the roots of $a\lambda^2 + b\lambda + c = 0$ are λ_1 and λ_2, then from notes in class we know that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are solutions of the ODE $a\ddot{y} + b\dot{y} + cy = 0$.

(a) Show that

$$\phi(t;\lambda_1,\lambda_2) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \quad \lambda_1 \neq \lambda_2,$$

is also a solution of the equation.

Solution. By taking $c_1 = -(\lambda_2 - \lambda_1)^{-1}$ and $c_2 = (\lambda_2 - \lambda_1)^{-1}$ in the general solution

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

we obtain ϕ.

(b) Calculate

$$\lim_{\lambda_2 \to \lambda_1} \phi(t;\lambda_1,\lambda_2)$$

and verify that the solution thus obtained is the second solution in the case of a repeated root.

Solution. Using L'Hôpital's Rule (taking the derivatives with respect to λ_2), we obtain

$$\lim_{\lambda_2 \to \lambda_1} \phi(t; \lambda_1, \lambda_2) = \lim_{\lambda_2 \to \lambda_1} \frac{te^{\lambda_2 t}}{1} = te^{\lambda_1 t},$$

which is exactly the second solution desired.

9. The displacement $x(t)$ of a spring is governed by the following equation:

$$\ddot{x} + 4\dot{x} + 4x = 0, \qquad x(0) = x_0 > 0, \quad \dot{x}(0) = v_0.$$

(a) Construct the solution to this problem.

Solution. Substituting $x = e^{\lambda t}$, we obtain

$$\lambda^2 + 4\lambda + 4 = 0 \qquad \Longrightarrow \qquad \lambda = -2,$$

so we have a double root. Therefore, our solutions are of the form $x = e^{-2t}(c_1 + c_2 t)$. Solving the first initial condition, we immediately have that

$$x(0) = c_1 = x_0.$$

Solving the second initial condition, we have

$$\begin{aligned} e^{-2t}(c_2) - 2e^{-t}(x_0 + c_2 t)\big|_{t=0} &= v_0 \\ c_2 - 2x_0 &= v_0 \\ x(t) &= e^{-2t}[x_0 + (v_0 + 2x_0)t]. \end{aligned}$$

(b) Show that $x(t_*) = 0$ if and only if $v_0/x_0 < -2$. In this case, how many times does the solution cross the t-axis? Interpret your results in terms of initial velocity and displacement.

Solution. The solution is zero whenever

$$\begin{aligned} x_0 + (v_0 + 2x_0)t &= 0 \\ t &= -\frac{x_0}{v_0 + 2x_0} > 0. \end{aligned} \tag{A}$$

For the last inequality in (A) to be true, we must have that the denominator is negative, so

$$\begin{aligned} v_0 + 2x_0 &< 0 \\ v_0/x_0 &< -2, \end{aligned}$$

as required. Note that there is only one solution for t in (A), and hence if the solution crosses the t-axis once, the solution crosses it *only* once. In terms of velocity and displacement, it says that if the initial velocity is positive or negative and too small compared with the initial displacement, the solution will never reach the t-axis.

10. Find the solution of

$$y^{(3)} - 3\ddot{y} + 4y = 0, \qquad y(0) = 4, \quad \dot{y}(0) = 4, \quad \ddot{y}(0) = 9.$$

Solution. Substituting $e^{\lambda t}$ into the above, we have

$$\lambda^3 - 3\lambda^2 + 4 = (\lambda^2 - \lambda - 2)(\lambda - 2) = (\lambda - 2)(\lambda + 1)(\lambda - 2).$$

Since $\lambda = 2$ is a double root, our solution is given by

$$\begin{aligned} y(t) &= c_1 e^{-t} + (c_2 + c_3 t)e^{2t}, \\ \dot{y}(t) &= -c_1 e^{-t} + 2(c_2 + c_3 t)e^{2t} + c_3 e^{2t}, \\ \ddot{y}(t) &= c_1 e^{-t} + 4(c_2 + c_3 t)e^{2t} + 4c_3 e^{2t}, \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} y(0) &= c_1 + c_2 = 4, \\ \dot{y}(0) &= -c_1 + 2c_2 + c_3 = 4, \\ \ddot{y}(0) &= c_1 + 4c_2 + 4c_3 = 9. \end{aligned}$$

Solving the above system using augmented matrices (you may use whatever method you wish), we have

$$\begin{matrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 4 \\ -1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 9 \end{pmatrix} \sim \begin{matrix} \mathrm{a} \\ \mathrm{d} = \mathrm{a} + \mathrm{b} \\ \mathrm{e} = \mathrm{b} + \mathrm{c} \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 3 & 1 & 8 \\ 0 & 6 & 5 & 13 \end{pmatrix} \sim \begin{matrix} \mathrm{a} \\ 5\mathrm{d} - \mathrm{e} \\ \mathrm{e} - 2\mathrm{d} \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 9 & 0 & 27 \\ 0 & 0 & 3 & -3 \end{pmatrix}.$$

Therefore, we have that $c_3 = -1$, $c_2 = 3$, $c_1 = 1$ and our solution is given by

$$y(t) = e^{-t} + (3 - t)e^{2t}.$$