Homework Set 2 Solutions

1. Write the general solution of the equation

$$\dot{y} = y^3 \sin t.$$

Solution.

$$\begin{aligned} \frac{dy}{y^3} &= \sin t \, dt \\ -\frac{1}{2y^2} &= -\cos t - \frac{C}{2} \\ y &= (C + 2\cos t)^{-1/2}. \end{aligned}$$

2. Write the solution of the equation

$$(t^2+1)\dot{y} = -(1+y^2), \qquad y(0) = 1.$$

Solution.

$$\begin{aligned} \frac{dy}{y^2+1} &= -\frac{dt}{t^2+1} \\ \tan^{-1} y &= -\tan^{-1} t - \tan^{-1} C \\ y &= -\tan(\tan^{-1} t + \tan^{-1} C) = -\frac{\tan(\tan^{-1} t) + \tan(\tan^{-1} C)}{1 - \tan(\tan^{-1} t)\tan(\tan^{-1} C)}, \end{aligned}$$

where we have used a trigonometric identity. Continuing to solve, we obtain

$$\begin{aligned} y &= -\frac{t+C}{1-Ct} \\ y(0) = 1 &= -C \\ y(t) &= -\frac{t+(-1)}{1-Ct} = \frac{1-t}{1+t}. \end{aligned}$$

3. **WITHOUT** solving the problem, determine the interval in t in which the solution of

$$(t-3)\dot{y} + (\log 4t)y = -t, \qquad y(1) = 3$$

is guaranteed to exist. Is the interval the same if the boundary condition is changed to

$$y(5) = 3?$$

Solution. Rewriting our equation in the standard form, we have

$$\dot{y} + \frac{\log 4t}{3-t} y = -\frac{t}{3-t}, \qquad p(t) = \frac{\log 4t}{3-t}, \quad g(t) = -\frac{t}{3-t}$$

Both p and g are undefined for $t = 3$ (the denominator). Also, p is undefined for $t \leq 0$ (the logarithm term) and If the boundary condition is given at $t = 1 < 3$, the solution is guaranteed to exist for $0 < t < 3$. If the boundary condition is given at $t = 5 > 3$, the solution is guaranteed to exist for $t > 3$.

4. Consider the equation

$$\dot{y} - y^5 = 0, \qquad y(0) = y_0 \neq 0.$$

(a) Write down the solution to the equation.

Solution.

$$\begin{aligned} \frac{dy}{y^5} &= dt \\ -\frac{1}{4y^4} &= t - \frac{C}{4} \\ y &= (C - 4t)^{-1/4} \\ y(0) = y_0 &= C^{-1/4} \\ y(t) &= \left(y_0^{-4} - 4t\right)^{-1/4}. \end{aligned}$$

(b) How does the interval of existence for the solution depend on y_0?

Solution. The solution exists only for when the quantity in parentheses is positive, so

$$\begin{aligned} 4t &< y_0^{-4} \\ t &< \frac{1}{4y_0^4}. \end{aligned}$$

5. Beginning at time $t = 0$, cigarette smoke containing 4% carbon monoxide (CO) is introduced into a room containing 50 m^3 of air at the rate of 0.01 m^3/min, and the well-circulated mixture leaves the room at the same rate.

(a) If $C(t)$ is the concentration of CO in the air at time t, show that

$$\frac{dC}{dt} + (2 \times 10^{-4})C = 8 \times 10^{-6}. \tag{2.1}$$

What is the initial condition?

Solution.

$$\begin{aligned} \frac{dC}{dt} &= \frac{\text{rate smoke added} - \text{rate smoke taken away}}{\text{volume of room}} \\ &= \frac{(4\%)\left(\frac{0.01\ \mathrm{m}^3}{\mathrm{min}}\right) - C\left(\frac{0.01\ \mathrm{m}^3}{\mathrm{min}}\right)}{50\ \mathrm{m}^3} = 8 \times 10^{-6} - (2 \times 10^{-4})C \end{aligned}$$

$$\frac{dC}{dt} + (2 \times 10^{-4})C = 8 \times 10^{-6},$$

as required. Initially, there is no smoke in the room, so $C(0) = 0$.

(b) Calculate $C(t)$.

Solution. The integrating factor for (2.1) is $e^{(2\times 10^{-4})t}$. Multiplying and integrating, we have

$$\begin{aligned}
\frac{d}{dt}\left(e^{(2\times 10^{-4})t}C\right) &= 8\times 10^{-6}e^{(2\times 10^{-4})t}\\
C &= e^{-(2\times 10^{-4})t}\left(\frac{8\times 10^{-6}}{2\times 10^{-4}}e^{(2\times 10^{-4})t} + A\right) = 0.04 + Ae^{-(2\times 10^{-4})t}\\
C(0) &= 0.04 + A = 0\\
C(t) &= 0.04\left[1 - e^{-(2\times 10^{-4})t}\right].
\end{aligned}$$

(c) Extended exposure to CO is harmful to the human body at levels of 0.012% or above. At what time does the concentration in the room reach that level?

Solution.

$$\begin{aligned}
C(t) = 0.04\left[1 - e^{-(2\times 10^{-4})t}\right] &= 1.2\times 10^{-4}\\
1 - e^{-(2\times 10^{-4})t} &= 0.03\\
e^{-(2\times 10^{-4})t} &= 0.97\\
-(2\times 10^{-4})t &= \log 0.97 = -0.030459207\\
t &= \frac{0.030459207}{2\times 10^{-4}} = 152.3 \text{ min.}
\end{aligned}$$

6. According to Newton's Law of Cooling, the rate of change of temperature θ of an object is proportional to the temperature differential between it and the outside environment (θ_∞).

(a) Explain in words why Newton's Law may be written as

$$\dot{\theta} = -k(\theta - \theta_\infty),$$

where k is a constant. Physically, what must the sign of k be?

Solution. The rate of change is $\dot{\theta}$, and choosing the constant of proportionality to be $-k$, we obtain the desired result. We expect that if the object is hotter than the exterior environment ($\theta - \theta_\infty > 0$), the object will cool ($\dot{\theta} < 0$). Therefore, we expect $k > 0$.

A pork loin initially at a temperature of 45° F is placed in a 425° F oven to cook. After 1 hr, the roast is 125° F.

(b) Calculate k for this example.

Solution.

$$\frac{d\theta}{\theta - \theta_\infty} = -k\,dt$$

$$\begin{aligned}
\log(\theta - \theta_\infty) &= -kt + \log C \\
\theta - 425 &= Ce^{-kt} \\
\theta(t) &= -380e^{-kt} + 425, \qquad \text{(A)} \\
\theta(1) = 125 &= -380e^{-k} + 425 \\
e^{-k} &= \frac{15}{19} \\
k &= \log 19 - \log 15 \approx 0.236.
\end{aligned}$$

(c) How much longer should the roast cook to be cooked medium (160° F)?

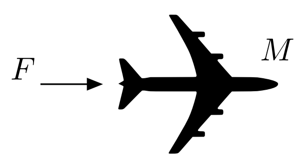
Solution. Using (A), we obtain

$$\begin{aligned}
\theta(t_m) &= 160 \\
-380e^{-kt} + 425 &= 160 \\
e^{-kt} &= \frac{265}{380} \\
t &= -\frac{\log(53/76)}{k} = \frac{0.360}{0.236} = 1.525,
\end{aligned}$$

so the roast should cook another 0.525 hr ≈ 31.5 min.

(d) Given your physical intuition, why is Newton's Law not a good model for this problem? (Usually Newton's Law is applied only to *surface* temperatures.)

Solution. In general, the temperature of a body will not be uniform; for instance, the center of the roast will be cold if only the outside of the roast is 160° F.



7. Consider an airplane of mass M moving horizontally through the air under a constant force F (see diagram). If the horizontal velocity is given by V, the frictional force is proportional to V^2 with constant k.

(a) Write a first-order ODE for V that describes this system. Be sure to explain the sign of each term.

Solution. The two forces in the problem are the forcing F, and the frictional force, which opposes the motion, so this force is given by $-kV^2$ (since positive V would produce a force to the left to oppose it). By Newton's Law, we have

$$Ma = M\dot{V} = F - kV^2. \qquad \text{(B)}$$

(b) Suppose that $F = k$, and the airplane starts from rest. Show that the solution is given by

$$V(t) = \frac{1 - e^{-2kt/M}}{1 + e^{-2kt/M}}. \qquad (2.2)$$

Solution. Letting $F = k$ in (B), we obtain

$$\begin{aligned} M\frac{dV}{dt} &= k - kV^2 = k(1-V)(1+V) \\ \frac{dV}{(1-V)(1+V)} &= \frac{k}{M}\,dt \\ dV\left[\frac{1}{2(1-V)} + \frac{1}{2(1+V)}\right] &= \frac{k}{M}\,dt \\ -\log(1-V) + \log(1+V) &= \frac{2kt}{M} + C, \end{aligned}$$

where we have used partial fractions. But $V(0) = 0$, so we have

$$\begin{aligned} -\log(1-0) + \log(1+0) &= \frac{2k(0)}{M} + C \\ C &= 0 \\ \log\left(\frac{1+V}{1-V}\right) &= \frac{2kt}{M} \\ \frac{1+V}{1-V} &= e^{2kt/M} \\ (1+V)e^{-2kt/M} &= 1 - V \\ V\left(1 + e^{-2kt/M}\right) &= 1 - e^{-2kt/M} \\ V(t) &= \frac{1 - e^{-2kt/M}}{1 + e^{-2kt/M}}. \end{aligned}$$

(c) What happens to the solution (2.2) as $t \to \infty$? Explain your answer physically.

Solution. As $t \to \infty$, the exponentials go to zero (because the exponents are negative), and we are left with $V = 1$. The velocity approaches a constant because the driving force is balanced with the frictional force.

8. Consider the differential equation

$$2\ddot{y} + 7\dot{y} + 3y = 0.$$

(a) Find the general solution. Describe the long-time behavior.

Solution. Substituting $y = e^{\lambda t}$, we obtain

$$\begin{aligned} 2\lambda^2 + 7\lambda + 3 &= 0 \\ (2\lambda + 1)(\lambda + 3) &= 0 \\ y(t) &= c_1 e^{-t/2} + c_2 e^{-3t}. \end{aligned}$$

Therefore, $y \to 0$ as $t \to \infty$.

(b) Calculate the specific solution for $y(0) = 3$, $\dot{y}(0) = -4$.

Solution. We must solve the following system:

$$
\begin{aligned}
c_1 + c_2 &= 3\\
-\frac{c_1}{2} - 3c_2 &= -4.
\end{aligned}
$$

The solution of this system may easily be found as $c_1 = 2$, $c_2 = 1$, so we have

$$y(t) = 2e^{-t/2} + e^{-3t}.$$

9. Write down *all* equations of the form $a\ddot{y} + b\dot{y} + cy = 0$ such that the solution y approaches a multiple of e^{-3t} as $t \to \infty$.

Solution. Substituting $y = e^{\lambda t}$, we obtain $a\lambda^2 + b\lambda + c = 0$. If the solution approaches a multiple of e^{-3t}, we see that the quadratic equation must have $\lambda = -3$ as a root and another root λ_2 which is *less than* $\lambda = -3$. (Otherwise, the solution would approach $e^{\lambda_2 t}$.) So we must have

$$
\begin{aligned}
a(\lambda+3)(\lambda-\lambda_2) &= 0\\
a\lambda^2 + a(3-\lambda_2)\lambda - 3a\lambda_2 &= 0, \qquad \lambda_2 < -3.
\end{aligned}
$$

10. Consider the following system of coupled first-order ODEs:

$$
\begin{aligned}
3\dot{x} + x - 4\dot{y} - 3y &= 0, &\text{(2.3a)}\\
-2x + \dot{y} + 4y &= 0. &\text{(2.3b)}
\end{aligned}
$$

(a) Eliminate x from the system to obtain a second-order ODE for y.

Solution. Solving (2.3b) for x, we obtain

$$x = -\frac{\dot{y} + 4y}{-2} = \frac{\dot{y}}{2} + 2y. \qquad \text{(C)}$$

Then substituting (C) into (2.3a), we have

$$
\begin{aligned}
3\frac{d}{dt}\left(\frac{\dot{y}}{2} + 2y\right) + \left(\frac{\dot{y}}{2} + 2y\right) - 4\dot{y} - 3y &= 0\\
3\ddot{y} + 12\dot{y} + \dot{y} + 4y - 8\dot{y} - 6y &= 0\\
3\ddot{y} + 5\dot{y} - 2y &= 0,
\end{aligned}
$$

where in the second line we have multiplied by 2.

(b) Show that the general solution for y is

$$y(t) = c_1 e^{t/3} + c_2 e^{-2t},$$

and find the corresponding general solution for x.

Solution. Substituting $y = e^{\lambda t}$, we obtain

$$\begin{aligned} 3\lambda^2 + 5\lambda - 2 &= 0 \\ (3\lambda - 1)(\lambda + 2) &= 0 \\ y(t) &= c_1 e^{t/3} + c_2 e^{-2t}. \end{aligned}$$

Then substituting this result into (C), we have

$$\begin{aligned} x &= \frac{1}{2}\left(\frac{c_1}{3} e^{t/3} - 2c_2 e^{-2t}\right) + 2\left(c_1 e^{t/3} + c_2 e^{-2t}\right) \\ x &= \frac{13c_1}{6} e^{t/3} + c_2 e^{-2t}. \end{aligned}$$