Supplemental Study Material Solutions

1. Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}.$$

(a) Find the general solution of this system.

Solution. Calculating the characteristic polynomial of the matrix, we have

$$p_A(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = \lambda^2 - 16 + 16 = \lambda^2.$$

Therefore, we have that $\lambda = 0$ is an eigenvalue of multiplicity 2. Solving $(A - 0I)\mathbf{z} = \mathbf{0}$, we have

$$\begin{pmatrix} 4 & -2 & 0 \\ 8 & -4 & 0 \end{pmatrix}.$$

Since the second row is a multiple of the first, we have $4x_1 - 2x_2 = 0$, so a characteristic eigenvector is $(1, 2)^T$. Therefore, one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To find the second solution, we solve $(A - 0I)\mathbf{z} = (1, 2)^T$:

$$\begin{pmatrix} 4 & -2 & 1 \\ 8 & -4 & 2 \end{pmatrix}. \tag{A}$$

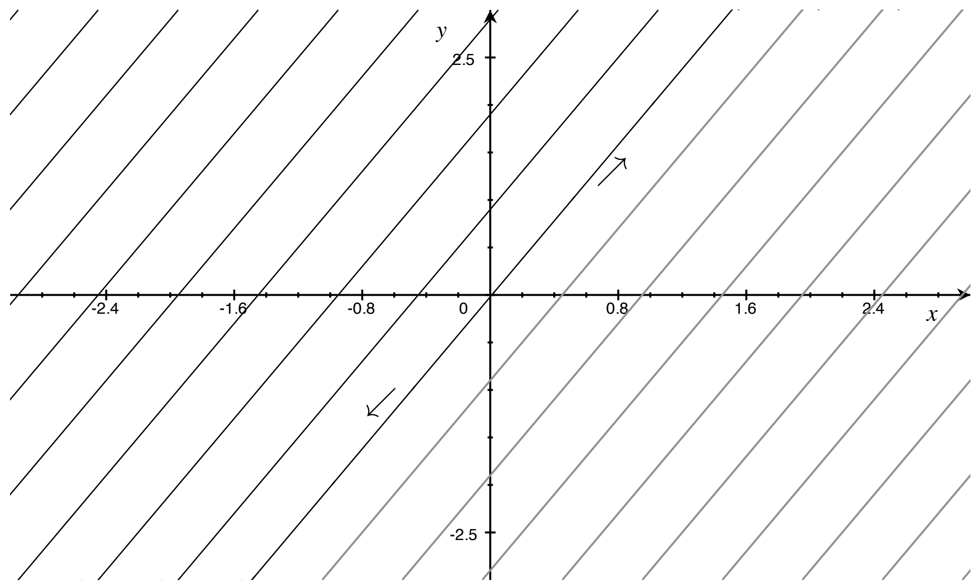
Since the second row is a multiple of the first, we have $4x_1 - 2x_2 = 1$, where x_2 is free. Therefore, a typical solution of (A) is given by $(1/4, 0)^T$, and so our second solution is

$$\mathbf{x}^{(2)} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}.$$

(If we had chosen any other typical solution of (A), the result would have differed by the above by some multiple of $\mathbf{x}^{(1)}$.) Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}\right].$$

(b) Sketch the phase plane for the system.



Phase plane for #1.

Solution.

(c) How do the solutions behave as $t \to \infty$?

Solution. In the special case where $c_2 = 0$, the solution stays fixed at whatever multiple of $(1,2)^T$ at which it began. Otherwise, it diverges algebraically by traveling along a line parallel to $(1,2)^T$.