Homework Set 10 Solutions

1. Consider the following matrix:

$$F_2 = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -2 & -2 \\ 6 & 6 & 6 \end{pmatrix}.$$

We wish to calculate the eigenvalues of this matrix *without* using the characteristic polynomial.

(a) Use facts about determinants to explain why $\lambda_1 = 0$.

Solution. We note that two rows are multiples of each other, and hence are linearly dependent. So the matrix is not invertible, its determinant is 0, and so the product of the eigenvalues must be 0. Hence $\lambda_1 = 0$.

(b) Use your answer to Homework Set 9, #6 to determine another eigenvalue.

Solution. We note that the sum of each column is 6, so by Homework Set 9, #6 we see that $\lambda_2 = 6$.

(c) Use facts about the trace to determine the third eigenvalue.

Solution. We note that $\operatorname{tr} F_2 = 6$, so the sum of the eigenvalues of the matrix must be 6. Therefore, the third eigenvalue must also be 0.

(d) Calculate the matrices S, Λ, and S^{-1} needed to diagonalize F_2.

Solution. From parts (b)–(d), we have that

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Solving for the eigenspaces, we have

$$(F_2 - 0I)\mathbf{z}_1 = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -2 & -2 \\ 6 & 6 & 6 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

The rows are all multiples of one another, so we see that $2x + 2y + 2z = 0$. Hence we have that $x = -y - z$. Therefore, the eigenspace is two dimensional consisting of vectors of the following form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the two vectors listed are columns of the S matrix. For the other eigenspace, we have

$$(F_2 - 6I)\mathbf{z}_2 = \begin{pmatrix} -4 & 2 & 2 \\ -2 & -8 & -2 \\ 6 & 6 & 0 \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

Row reducing, we have

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} -4 & 2 & 2 & 0 \\ -2 & -8 & -2 & 0 \\ 6 & 6 & 0 & 0 \end{pmatrix} \sim \begin{matrix} \text{c} - 3\text{a} \\ -2\text{b} + \text{a} \\ \text{a} + \text{b} + \text{c} \end{matrix} \begin{pmatrix} 18 & 0 & -6 & 0 \\ 0 & 18 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{matrix} 18x - 6z = 0 \\ 18y + 6z = 0 \\ z \text{ free} \end{matrix}$$

Choosing $z = 3$, then $x = 1$, $y = -1$, and a typical eigenvector is $(1, -1, 3)^T$. Thus we have

$$S = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}.$$

To find the inverse, we use row reduction to obtain

$$\begin{matrix} \text{a} \\ \text{b} \\ \text{c} \end{matrix} \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \sim \begin{matrix} \text{d} = \text{c} + 2\text{b} + \text{a} \\ -(\text{b} + \text{a}) \\ \text{e} = \text{c} + \text{b} + \text{a} \end{matrix} \begin{pmatrix} 1 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{matrix} \text{d} - 2\text{e}/3 \\ \\ \text{e}/3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1/3 & 4/3 & 1/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

Therefore, we have that

$$S^{-1} = \begin{pmatrix} 1/3 & 4/3 & 1/3 \\ -1 & -1 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

2. Consider the following matrix and vector:

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$

(a) Find the eigenvalues of A.

Solution. Calculating the characteristic polynomial, we have

$$\begin{vmatrix} (4-\lambda) & 3 \\ 1 & (2-\lambda) \end{vmatrix} = (4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0.$$

Therefore, we have that $\lambda_1 = 1$ and $\lambda_2 = 5$.

(b) Find the corresponding eigenvectors of A.

Solution. Solving for the eigenspaces, we have

$$(A - I)\mathbf{z}_1 = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

The rows are multiples of one another, so we see that $x + y = 0$, so a typical eigenvector is $(1, -1)^T$. Similarly, we have

$$(A - 5I)\mathbf{z}_2 = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

The rows are multiples of one another, so we see that $x - 3y = 0$, so a typical eigenvector is $(3, 1)^T$.

(c) Write the spectral decomposition $A = S\Lambda S^{-1}$.

Solution. Using our answers from (a) and (b), we have

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad S^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/4 & -3/4 \\ 1/4 & 1/4 \end{pmatrix}.$$

(d) Calculate $[A\mathbf{x}]_Z$ and $\Lambda[\mathbf{x}]_Z$. Verify that they are equal.

Solution. To calculate the coordinates in the Z basis, we need the transition matrix $_{Z\leftarrow}P_E$, which is simply the inverse of the matrix of eigenvectors. Thus the transition matrix is S^{-1}. So we have

$$[A\mathbf{x}]_Z = S^{-1}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1/4 & -3/4 \\ 1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 11 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix},$$

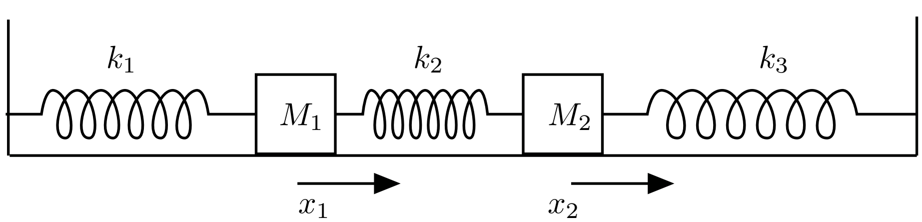
$$\Lambda[\mathbf{x}]_Z = \Lambda\begin{pmatrix} 1/4 & -3/4 \\ 1/4 & 1/4 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}.$$

(e) Use your answer to part (c) to calculate $A^{10}\mathbf{x}$.

Solution.

$$A^{10}\mathbf{x} = (S\Lambda S^{-1})^{10}\mathbf{x} = S\Lambda^{10}S^{-1}\mathbf{x} = S\Lambda^{10}[\mathbf{x}]_Z = S\begin{pmatrix} 1 & 0 \\ 0 & 5^{10} \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 5^{10} \end{pmatrix} = \begin{pmatrix} -4 + 3 \cdot 5^{10} \\ 4 + 5^{10} \end{pmatrix} = \begin{pmatrix} 29296871 \\ 9765629 \end{pmatrix}$$



3. Consider the diagram above, which shows two bobs (mass M_j) connected by three springs to each other and external walls. Here x_j is the position of bob j measured from its equilibrium position (not necessarily indicated in the diagram).

   (a) Show that the system governing the motion of each bob is given by

$$M_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1), \tag{10.1a}$$
$$M_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1). \tag{10.1b}$$

   Be sure to clearly explain the sign of each term.

Solution. The left-hand side of each equation is the force on the bob. As bob 1 moves to the left ($x_1 < 0$), the first spring pushes in the positive x_1 direction, so this term must be $-k_1x_1$. The second spring can be stretched by either bob, so its spring force must be proportional to the distance between them, or $x_2 - x_1$. Suppose that bob 1 moves to the left while bob 2 remains fixed, so $x_2 - x_1 > 0$. In that case, the second spring would pull bob 1 to the right, so that term must be $k_2(x_2 - x_1)$. The same set of circumstances would pull bob 2 to the right, so that term must be $-k_2(x_2 - x_1)$ in (10.1b). Finally, if bob 2 moves to the right ($x_2 > 0$), then the third spring will exert a force in the negative x_2 direction, so this term must be $-k_3x_2$.

   (b) By introducing the velocity v_j of each bob, write (10.1) as a system of four coupled *first-order* ODEs.

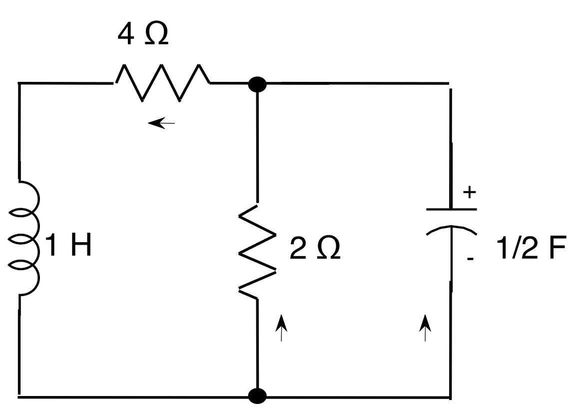
Solution. By letting

$$\dot{x}_1 = v_1,$$
$$\dot{x}_2 = v_2,$$

in (10.1), we obtain

$$M_1\dot{v}_1 = -k_1x_1 + k_2(x_2 - x_1),$$
$$M_2\dot{v}_2 = -k_3x_2 - k_2(x_2 - x_1).$$

These four first-order ODEs form the system we seek.



4. Consider the circuit shown above.

The current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$\dot{I} = -4I - V, \tag{10.2a}$$
$$\dot{V} = 2I - V. \tag{10.2b}$$

(b) Combine (10.2) to form a single second-order differential equation for V.

Solution. Taking the derivative of (10.2b) with respect to t and adding it to twice (10.2a), we have

$$\begin{aligned} 2\dot{I} + \ddot{V} &= -8I - 2V + 2\dot{I} - \dot{V} \\ \ddot{V} + \dot{V} + 2V + 8\left(\frac{\dot{V}+V}{2}\right) &= 0 \\ \ddot{V} + 5\dot{V} + 6V &= 0, \end{aligned} \tag{A}$$

where we have used the definition of I in (10.2b).

(c) Solve (10.2) for V, then substitute into (10.2b) to find the solution for I.

Solution. Substituting $V = e^{\lambda t}$ into (A), we obtain

$$\begin{aligned} \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \qquad &\Longrightarrow \qquad \lambda_1 = -3, \quad \lambda_2 = -2, \\ V(t) &= c_1 e^{-3t} + c_2 e^{-2t}. \end{aligned}$$

Then substituting this result into (10.2b), we have

$$\begin{aligned} -3c_1 e^{-3t} - 2c_2 e^{-2t} &= 2I - (c_1 e^{-3t} + c_2 e^{-2t}) \\ I(t) &= \frac{-2c_1 e^{-3t} - c_2 e^{-2t}}{2}. \end{aligned}$$

(d) Find the solution for V and I if $V(0) = 0$, $I(0) = 3$.

Solution.

$$\begin{aligned} V(0) = c_1 + c_2 = 0 \qquad &\Longrightarrow \qquad c_2 = -c_1 \\ I(0) = -\frac{2c_1 + c_2}{2} = -\frac{c_1}{2} = 3 \qquad &\Longrightarrow \qquad c_1 = -6, \quad c_2 = 6 \\ V(t) &= 6(e^{-2t} - e^{-3t}), \\ I(t) &= \frac{12e^{-3t} - 6e^{-2t}}{2} = 3(2e^{-3t} - e^{-2t}). \end{aligned}$$

5. Consider the vectors

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 7 \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}.$$

(a) Calculate the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

Solution.

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = \begin{vmatrix} t & 2e^{-t} \\ 7 & e^{-t} \end{vmatrix} = (t-14)e^{-t}.$$

(b) Where are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?

Solution. The solutions are linearly independent wherever the Wronskian is not zero, *i.e.*, where $t \neq 14$.

(c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?

Solution. Since the solutions are not linearly independent at $t = 14$, we expect that at least one of the coefficients of the system will be discontinuous at $t = 14$.

(d) Show (by deriving the coefficients of A, **NOT** by direct substitution) that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are the solutions of

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad A = \frac{1}{t-14}\begin{pmatrix} 15 & -2(t+1) \\ 7 & -t \end{pmatrix},$$

and hence verify your answer to (c).

Solution. If we multiply out $\dot{\mathbf{x}} = A\mathbf{x}$, we get

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2, \end{aligned}$$

and our task is to find the a_{ij}. Since $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are both solutions of $\dot{\mathbf{x}} = A\mathbf{x}$, we may substitute in to obtain the following set of four equations:

$$\dot{x}_1^{(1)} = 1 = a_{11}t + a_{12}7 \qquad \dot{x}_2^{(1)} = 0 = a_{21}t + a_{22}7$$
$$\dot{x}_1^{(2)} = -2e^{-t} = a_{11}(2e^{-t}) + a_{12}e^{-t} \qquad \dot{x}_2^{(2)} = -e^{-t} = a_{21}(2e^{-t}) + a_{22}e^{-t}$$

Actually, the two on the left form a system in a_{11} and a_{12}; solving them together yields

$$15 = (t-14)a_{11}, \quad a_{12} = -\frac{2(t+1)}{t-14}.$$

Similarly, the two equations on the left form a system in a_{21} and a_{22}; solving them together yields

$$7 = (t-14)a_{21}, \quad a_{22} = -\frac{t}{t-14}.$$

Therefore, we obtain the desired result. Note that all the coefficients are discontinuous at $t = 14$, as surmised.

6. Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} -4 & 4 & -2 \\ -1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}\mathbf{x}.$$

(a) Show that the eigenvalues for this matrix are $\lambda_1 = 0$, $\lambda_2 = -2$, and $\lambda_3 = -1$.

Solution. Calculating the characteristic polynomial, we have

$$\begin{aligned}\begin{vmatrix} -4-\lambda & 4 & -2 \\ -1 & 1-\lambda & -1 \\ 2 & -2 & -\lambda \end{vmatrix} &= 2\begin{vmatrix} 4 & -2 \\ 1-\lambda & -1 \end{vmatrix} + 2\begin{vmatrix} -4-\lambda & -2 \\ -1 & -1 \end{vmatrix} - \lambda\begin{vmatrix} -4-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} \\ &= 2[-4+2(1-\lambda)] + 2(4+\lambda-2) - \lambda[-(4+\lambda)(1-\lambda)+4] \\ &= -\lambda^3 - 3\lambda^2 - 2\lambda = -\lambda(\lambda+2)(\lambda+1) = 0.\end{aligned}$$

Thus we have the desired result.

(b) Find the general solution $\mathbf{x}(t)$ of this system.

Solution. Now we must calculate the eigenvectors corresponding to the eigenvalues. Solving for the first eigenspace, we obtain

$$(A - 0I)\mathbf{z}_1 = \begin{pmatrix} -4 & 4 & -2 \\ -1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

We note that the first column plus the second column is the zero vector, so $1\mathbf{a}_1 + 1\mathbf{a}_2 + 0\mathbf{a}_3 = \mathbf{0}$. But the left-hand side is the definition of matrix-vector multiplication, so we have that a typical eigenvector is $\mathbf{z}_1 = (1, 1, 0)^T$. Solving for the second eigenspace, we obtain

$$B = (A + 2I)\mathbf{z}_2 = \begin{pmatrix} -2 & 4 & -2 \\ -1 & 3 & -1 \\ 2 & -2 & 2 \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

(Note that we have defined a new matrix B.) We note that the first column minus the third column is the zero vector, so $1\mathbf{b}_1 + 0\mathbf{b}_2 - 1\mathbf{b}_3 = \mathbf{0}$. But the left-hand side is the definition of matrix-vector multiplication, so we have that a typical eigenvector is $\mathbf{z}_2 = (1, 0, -1)^T$. Solving for the third eigenspace, we obtain

$$C = (A + I)\mathbf{z}_3 = \begin{pmatrix} -3 & 4 & -2 \\ -1 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{z}_3 = \mathbf{0}.$$

We note that the second column plus two times the third column is the zero vector, so $0\mathbf{c}_1 + 1\mathbf{b}_2 + 2\mathbf{c}_3 = \mathbf{0}$. But the left-hand side is the definition of matrix-vector multiplication, so we have that a typical eigenvector is $\mathbf{z}_3 = (0, 1, 2)^T$. Therefore, our solution is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(c) What happens to the solution as $t \to \infty$?

Solution. As $t \to \infty$, the exponentials decay to zero and we are left with

$$\lim_{t\to\infty} \mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

7. Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \mathbf{x}. \tag{10.3}$$

(a) Show that the eigenvectors for this system are $\mathbf{z}_1 = (1, 2)^T$, $\mathbf{z}_2 = (1, -1)^T$.

Solution. Calculating the characteristic polynomial, we have

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2 - 8 = (\lambda - 5)(\lambda + 1) = 0.$$

Therefore, the eigenvalues of the matrix are $\lambda_1 = 5$ and $\lambda_2 = -1$. Solving for the first eigenspace, we obtain

$$(A - 5I)\mathbf{z}_1 = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

We note that the first column plus twice the second column is the zero vector. But this is the definition of matrix-vector multiplication, so we have that a typical eigenvector is $\mathbf{z}_1 = (1, 2)^T$. Solving for the second eigenspace, we obtain

$$(A + I)\mathbf{z}_2 = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

We note that the first column minus the second column is the zero vector. But this is the definition of matrix-vector multiplication, so we have that a typical eigenvector is $\mathbf{z}_2 = (1, -1)^T$.

(b) Find the general solution $\mathbf{x}(t)$ of this system.

Solution.

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{B}$$

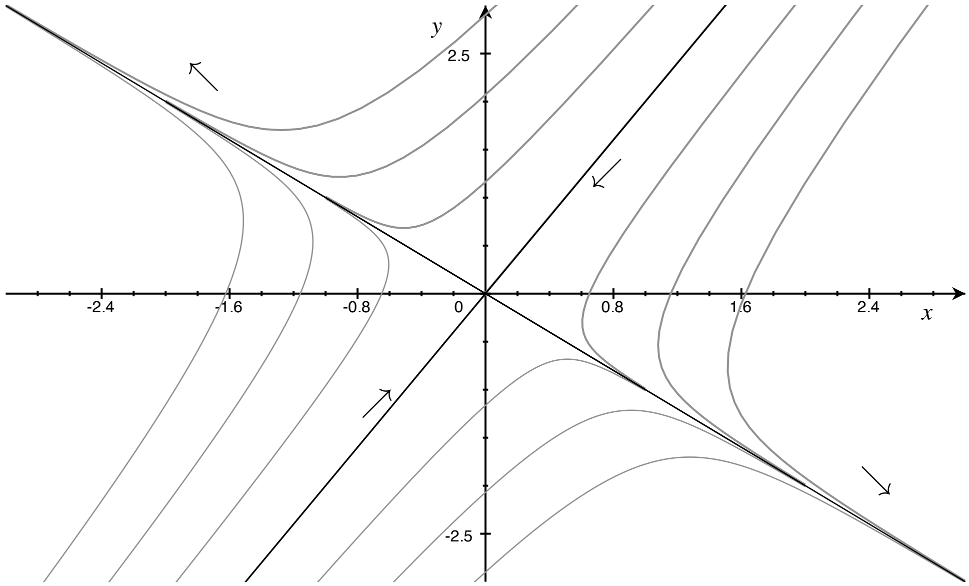
(c) Sketch the phase plane for this system. Classify the fixed point.

Solution. See below. The origin is a saddle point.

(d) Find the solution of the initial-value problem given by (10.3) and $\mathbf{x}(0) = (0, 3)^T$.

Solution. Substituting $t = 0$ into (B), we obtain

$$\begin{aligned} c_1 + c_2 &= 0 \\ 2c_1 - c_2 &= 3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned} \quad \Longrightarrow \quad \mathbf{x}(t) = \begin{pmatrix} e^{5t} - e^{-t} \\ 2e^{5t} + e^{-t} \end{pmatrix}.$$



8. Consider the matrix

$$A = \begin{pmatrix} 2 & \gamma \\ 2 & 2 \end{pmatrix} \mathbf{x}.$$

(a) Find the eigenvalues and eigenvectors of A.

Solution. Calculating the characteristic polynomial, we have

$$\begin{vmatrix} 2-\lambda & \gamma \\ 2 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 2\gamma = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(4-2\gamma)}}{2} = 2 \pm \sqrt{2\gamma}.$$

Solving for the first eigenspace, we obtain

$$(A - (2+\sqrt{2\gamma})I)\mathbf{z}_1 = \begin{pmatrix} -\sqrt{2\gamma} & \gamma \\ 2 & -\sqrt{2\gamma} \end{pmatrix} \mathbf{z}_1 = \mathbf{0}.$$

We note that $\sqrt{\gamma/2}$ times the first column plus the second column is the zero vector. But this is the definition of matrix-vector multiplication, so we have that a typical eigenvector is $\mathbf{z}_1 = (\sqrt{\gamma/2}, 1)^T$. Solving for the second eigenspace, we obtain

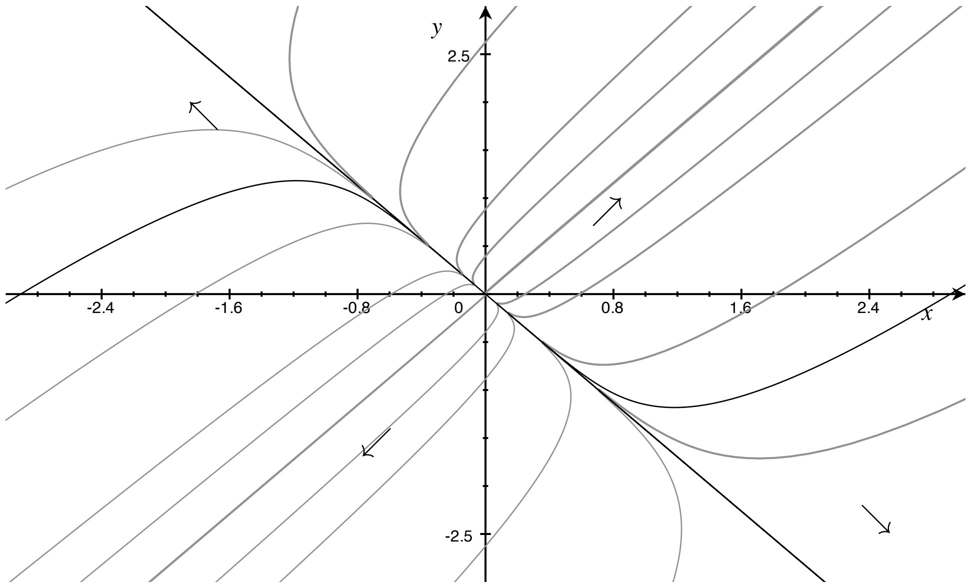
$$(A - (2-\sqrt{2\gamma})I)\mathbf{z}_2 = \begin{pmatrix} \sqrt{2\gamma} & \gamma \\ 2 & \sqrt{2\gamma} \end{pmatrix} \mathbf{z}_2 = \mathbf{0}.$$

We note that $\sqrt{\gamma/2}$ times the first column minus the second column is the zero vector. But this is the definition of matrix-vector multiplication, so we have that a typical eigenvector is $\mathbf{z}_2 = (\sqrt{\gamma/2}, -1)^T$.

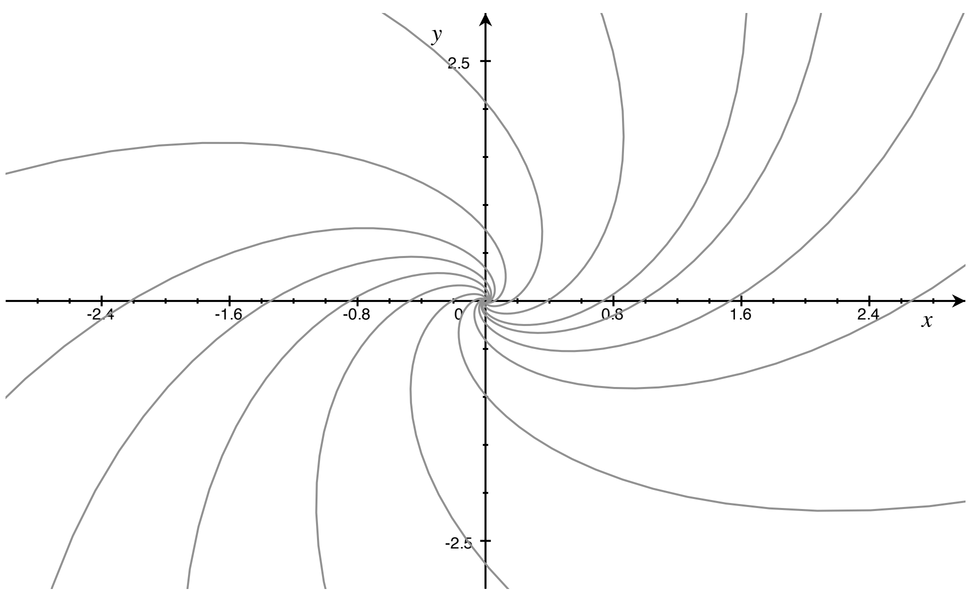
(b) Why will the qualitative nature of the phase plane change when γ passes through the critical value zero?

Solution. When $\gamma > 0$, the eigenvalues and eigenvectors are real. When $\gamma < 0$, they are complex.

(c) Sketch phase planes for γ less than and greater than zero. Classify the fixed point for $\gamma < 0$.



Phase plane for $\gamma = 1$.



Phase plane for $\gamma = -1$. The origin is an unstable spiral.

(d) Find the general solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$.

Solution.

$$\mathbf{x}(t) = c_+ e^{(2+\sqrt{2\gamma})t}\begin{pmatrix}\sqrt{\gamma/2}\\ 1\end{pmatrix} + c_+ e^{(2-\sqrt{2\gamma})t}\begin{pmatrix}-\sqrt{\gamma/2}\\ -1\end{pmatrix}.$$

9. We reconsider the circuit in #4.

(a) Write the system (10.2) in matrix-vector form.

Solution. Referring to (10.2), we have

$$\dot{\mathbf{x}} = B\mathbf{x}, \qquad B = \begin{pmatrix}-4 & -1\\ 2 & -1\end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix}I\\ V\end{pmatrix}.$$

(b) Solve the system for I and V using eigenvalues and eigenvectors. Confirm that your answer agrees with #4(c).

Solution. Calculating the characteristic polynomial, we have

$$\begin{vmatrix}-4-\lambda & -1\\ 2 & -1-\lambda\end{vmatrix} = 4 + 5\lambda + \lambda^2 + 2 = \lambda^2 + 5\lambda + 6$$
$$(\lambda+3)(\lambda+2) = 0$$
$$\lambda_1 = -3, \qquad \lambda_2 = -2.$$

Solving for the first eigenspace, we obtain

$$(A - (-3)I)\mathbf{z}_1 = \begin{pmatrix}-1 & -1\\ 2 & 2\end{pmatrix}\mathbf{z}_1 = \mathbf{0}.$$

We note that the second row is -2 times the first, so the equations are redundant. Therefore, we have that $2x + 2y = 0$, so taking $y = 1$, we have a typical eigenvector of $\mathbf{z}_1 = (-1, 1)^T$. Solving for the second eigenspace, we obtain

$$(A - (-2)I)\mathbf{z}_1 = \begin{pmatrix}-2 & -1\\ 2 & 1\end{pmatrix}\mathbf{z}_2 = \mathbf{0}.$$

We note that the second row is -1 times the first, so the equations are redundant. Therefore, we have that $2x + y = 0$, so taking $y = 1$, we have a typical eigenvector of $\mathbf{z}_2 = (-1/2, 1)^T$. Then using these results, we have

$$\begin{pmatrix}I\\ V\end{pmatrix} = c_1 e^{-3t}\begin{pmatrix}-1\\ 1\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}-1/2\\ 1\end{pmatrix}, \tag{C}$$

which matches the solutions in #4(c). (If we had chosen different eigenvectors, we would just adjust the arbitrary constants to match.)

(c) Does the solution depend on the initial data as $t \to \infty$?

Solution. No, because both eigenvalues are negative. Hence in this case $e^{\lambda t} \to 0$ as $t \to \infty$ for all sets of initial data.

(d) Find the solution for V and I if $V(0) = 0$, $I(0) = 3$. Confirm that your answer agrees with #4(d).

Solution. Substituting $t = 0$ into (C), we have

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{aligned} c_1 &= -c_2 \\ -(-c_2) - \frac{c_2}{2} &= 3 \end{aligned}$$

$$c_2 = 6, \qquad c_1 = -6$$

$$\begin{pmatrix} I \\ V \end{pmatrix} = -6e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 6e^{-2t} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(2e^{-3t} - e^{-2t}) \\ 6(e^{-2t} - e^{-3t}) \end{pmatrix},$$

which matches with the answer to #4(d).