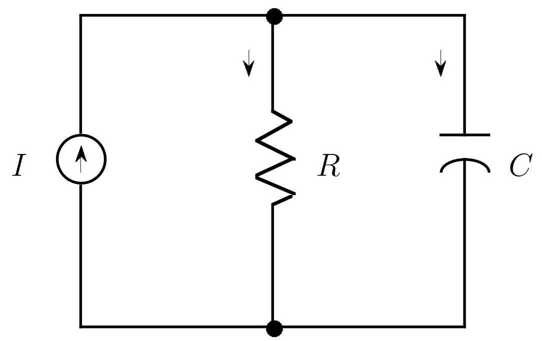
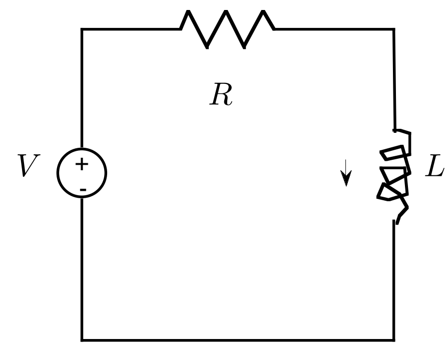
Homework Set 1 Solutions





1. Determine by inspection two solutions of

$$(y')^2 + y^2 = 1. \tag{1.1}$$

Solution. The form (1.1) reminds one of the trigonometric identity

$$\cos^2 x + \sin^2 x = 1.$$

Since

$$\frac{d(\sin x)}{dx} = \cos x,$$

we see that $y = \sin x$ is one solution of (1.1). Similarly, since

$$\frac{d(\cos x)}{dx} = -\sin x,$$

we see that $y = \cos x$ is another solution of (1.1).

2. Consider the driven RC circuit shown above left. The current through the resistor is given by $I_R = V/R$, where V is the voltage, and the current through the capacitor is given by $I_C = C\dot{V}$. Let the driving current $I = A\cos\omega t$, where A is the constant amplitude and ω the constant frequency. (The directions of the currents are indicated by the arrows.) The sum of the currents into the node above the resistor must be zero; use this fact to obtain the differential equation

$$\dot{V} + \frac{1}{RC}V = \frac{A\cos\omega t}{C}.$$

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Solution. By the directions of the arrows, the current flowing out of the node is given by $I_R + I_C$, while the the current flowing into the node is the imposed current $A\cos\omega t$. Since current in must equal out (so the net current is zero), we have that

$$C\dot{V} + \frac{1}{R}V = A\cos\omega t,$$

and dividing by C gives the desired result.

3. Consider the driven RL circuit shown above right. The voltage through the inductor is given by $V_L = L\dot{I}$, where I is the current. Let the driving voltage $V = A\cos\omega t$, where A is the constant amplitude and ω the constant frequency. The sum of the voltage around this loop must be zero; use this fact to obtain the differential equation

$$\dot{I} + \frac{R}{L}I = \frac{A\cos\omega t}{L}.$$

Solution. Since the sign of the voltage drop is the same as the sign of the first sign as the current goes around, we see that the imposed voltage is negative. Therefore, we have $V_L + V_R - A\cos\omega t = 0$. From problem 6, we see that $V_R = IR$, so we have

$$L\dot{I} + RI = A\cos\omega t,$$

from which the desired result immediately follows.

4. A ball of mass M is driven horizontally through a fluid by a constant force F. The frictional force is proportional to the horizontal velocity V with proportionality constant k. Show that the governing equation for V is given by

$$M\dot{V} + kV = F.$$

Be sure to explain the sign of each term.

Solution. The two forces in the problem are the forcing F, and the frictional force, which opposes the motion, so this force is given by $-kV$ (since positive V would produce a force to the left to oppose it). By Newton's Law, we have

$$Ma = M\dot{V} = F - kV,$$

which upon rearrangement becomes the desired result.

5. Consider the equation

$$4\dot{y} = -2y + 3. \tag{1.2}$$

(a) Show by direct substitution that a solution of the form

$$y(t) = A + Be^{\lambda t}. \tag{1.3}$$

satisfies (1.2). Calculate exact values for as many of the constants $\{A, B, \lambda\}$ as possible. Which terms in equation (1.2) help you determine which constants?

Solution. Substituting (1.3) into (1.2), we obtain

$$\begin{aligned} 4(\lambda B e^{\lambda t}) &= -2(A + Be^{\lambda t}) + 3 \\ 2B(2\lambda + 1)e^{\lambda t} &= -2A + 3. \end{aligned}$$

Matching coefficients, we see that (1.3) is a solution as long as $\lambda = -1/2$ and $A = 3/2$. We are unable to determine B at this time. The $4\dot{y}$ and $-2y$ terms determined λ, and the right-hand side of (1.2) determined A.

Now suppose that in addition to (1.2),

$$y(0) = 1. \tag{1.4}$$

(b) Use (1.4) to calculate exact values for all the constants in the proposed solution (1.3).

Solution. From part (a), we have that our solution is

$$y(t) = \frac{3}{2} + Be^{-t/2}.$$

Substituting $t = 0$ into the above, we have

$$y(0) = \frac{3}{2} + B = 1,$$

so $B = -1/2$ and our final solution is

$$y(t) = \frac{3 - e^{-t/2}}{2}.$$

6. Solve the initial-value problem

$$y' + (\tan x)y = \cos^2 x, \qquad y(0) = -1,$$

and identify the largest interval over which it is defined.

Solution. Since $p(x) = \tan x$, the integrating factor is

$$\mu = \exp\left(\int \frac{\sin x}{\cos x}\, dx\right) = \exp(-\log\cos x) = (\cos x)^{-1}.$$

Multiplying by μ and integrating, we have

$$\begin{aligned} \frac{y'}{\cos x} + \frac{\sin x}{\cos^2 x} y = \frac{d}{dx}\left(\frac{y}{\cos x}\right) &= \cos x \\ \frac{y}{\cos x} &= \sin x + C \\ y &= \sin x \cos x + C\cos x \\ y(0) &= C = -1 \\ y(x) &= (\sin x - 1)\cos x. \end{aligned}$$

This solution is defined everywhere, even though $\tan x$ is discontinuous whenever $\cos x = 0$, or $x = (2m+1)\pi/2$.

7. Consider the differential equation

$$\dot{y} - 3y = e^{-t}, \qquad y(0) = y_0.$$

Solution. Since $p(t) = -3$, the integrating factor is e^{-3t}. Multiplying by this factor and integrating, we have

$$\begin{aligned} e^{-3t}\dot{y} - 3e^{-3t}y &= e^{-4t} \\ \frac{d(e^{-3t}y)}{dt} &= e^{-4t} \\ e^{-3t}y &= -\frac{e^{-4t}}{4} + C \\ y(t) &= -\frac{e^{-t}}{4} + Ce^{3t} \\ y(0) = C - \frac{1}{4} &= y_0 \\ C &= y_0 + \frac{1}{4} \\ y(t) &= -\frac{e^{-t}}{4} + \left(y_0 + \frac{1}{4}\right)e^{3t}. \end{aligned}$$

(b) Describe how the long-time behavior of y varies with y_0.

Solution. As $t \to \infty$, $y(t)$ becomes exponentially large, and the sign of $y(t)$ is the same as the sign of $y_0 + 1/4$. Therefore, we have

$$\lim_{t\to\infty} y(t) = \begin{cases} \infty, & y_0 > -1/4, \\ -\infty, & y_0 < -1/4. \end{cases}$$

(c) Find the critical value of y_0 which separates the two types of behaviors.

Solution. From part (b), we see that the critical value is $y_0 = -1/4$.

(d) Describe the long-time behavior of y for that specific value of y_0.

Solution. For $y_0 = -1/4$, the solution is $-e^{-t}/4$, which goes to 0 as $t \to \infty$.

8. Solve the differential equation

$$t\dot{y} - 4(t+1)y = e^{4t}, \qquad y(1) = 3.$$

Solution. Dividing by t to obtain the standard form, we have

$$\dot{y} - 4\left(1 + \frac{1}{t}\right)y = \frac{e^{4t}}{t},$$

so the integrating factor is

$$\exp\left(-4\int 1+\frac{1}{t}\,dt\right) = \exp\left(-4(t+\log t)\right) = \frac{e^{-4t}}{t^4}.$$

Multiplying and integrating, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{e^{-4t}}{t^4}y\right) &= t^{-5}\\ y(t) &= t^4e^{4t}\left(-\frac{1}{4t^4}+C\right) = e^{4t}(Ct^4-1/4)\\ y(1) &= e^4(C-1/4) = 3\\ C &= 3e^{-4}+\frac{1}{4}\\ y(t) &= e^{4t}\left[\left(3e^{-4}+\frac{1}{4}\right)t^4-\frac{1}{4}\right].\end{aligned}$$

9. Consider the differential equation

$$\dot{y}+\frac{\alpha}{t}y = t.$$

(a) Find the general solution for any α. Be sure to consider the special case where $\alpha = -2$.

Solution. The integrating factor is

$$\exp\left(\alpha\int\frac{1}{t}\,dt\right) = \exp(\alpha\log t) = t^\alpha,$$

so we have

$$\begin{aligned}\frac{d}{dt}\left(t^\alpha y\right) &= t^{\alpha+1}\\ y &= t^{-\alpha}\left(\frac{t^{\alpha+2}}{\alpha+2}+C\right) = \frac{t^2}{\alpha+2}+Ct^{-\alpha}, \quad \alpha\neq -2.\end{aligned}$$

If $\alpha = -2$, we have

$$\begin{aligned}\frac{d}{dt}\left(t^{-2}y\right) &= \frac{1}{t}\\ y &= t^2\left(\log t + C\right).\end{aligned}$$

(b) For what values of α does the solution stay bounded as $t\to\infty$?

Solution. Because of the t^2 term, the solution is unbounded for all α.

(c) Where is your solution discontinuous?

Solution. For $\alpha > 0$ and $\alpha = -2$, the solution is discontinuous at $t = 0$.

10. In #3, you showed that for the driven RL circuit shown above, the governing equation for the current I is given by

$$\dot{I} + \frac{R}{L} I = \frac{A \cos \omega t}{L}.$$

(a) If the initial current is zero, $R = 2\,\Omega$, $A = 1$ V, and $L = 2$ H, show that

$$I(t) = \frac{\cos \omega t - e^{-t} + \omega \sin \omega t}{2(\omega^2 + 1)}. \tag{1.5}$$

Do the initial conditions matter as $t \to \infty$? (You may do complicated integrals using a computer, but you must provide a printout.)

Solution. Making these substitutions, we have

$$\dot{I} + I = \frac{\cos \omega t}{2}, \qquad I(0) = 0.$$

Since $p(t) = 1$, the integrating factor is e^t and so

$$\begin{aligned} \frac{d}{dt}\left(e^t I\right) &= \frac{1}{2} e^t \cos \omega t \\ e^t I &= \frac{K_c}{2} + k, \qquad K_c \equiv \int e^t \cos \omega t \, dt, \end{aligned} \tag{A}$$

where k is a constant. To calculate K_c we integrate by parts twice to obtain

$$\begin{aligned} K_c &= e^t \cos \omega t + \omega \int e^t \sin \omega t \, dt \\ &= e^t \cos \omega t + \omega \left(e^t \sin \omega t - \omega \int e^{\lambda t} \cos \omega t \, dt \right) \\ &= e^t (\cos \omega t + \omega \sin \omega t) - \omega^2 K_c, \end{aligned}$$

where we have used the definition of K_c in (A). Continuing to simplify, we have

$$\begin{aligned} (1 + \omega^2) K_c &= e^t \cos \omega t + \omega e^t \sin \omega t \\ K_c &= \frac{e^t \cos \omega t + \omega e^t \sin \omega t}{\omega^2 + 1} \\ I = e^{-t} \left(\frac{K_c}{2} + k \right) &= \frac{\cos \omega t + \omega \sin \omega t}{2(\omega^2 + 1)} + k e^{-t}, \end{aligned}$$

where k is a constant we will use to solve the initial data. But note that for any k, as $t \to \infty$ that term goes to zero while the others oscillate. So the initial conditions don't make any difference to the *steady state* the solution approaches as $t \to \infty$.

Satisfying the initial condition, we have

$$\begin{aligned} I(0) = 0 &= \frac{1}{2(\omega^2+1)} + k \\ k &= -\frac{1}{2(\omega^2+1)}, \\ I(t) &= \frac{\cos\omega t - e^{-t} + \omega\sin\omega t}{2(\omega^2+1)}, \end{aligned}$$

as required.

Often when studying electrical circuits the imposed current or voltage is turned on or off suddenly. In the equation

$$\dot{y} + p(t)y = g(t),$$

this corresponds to $g(t)$ being discontinuous. When this occurs, we solve the problem in each interval where g is continuous, and then require that y be continuous where the intervals join together.

Consider the circuit above, but now suppose that we shut the voltage off after one cycle, so we have

$$V = \begin{cases} A\cos\omega t, & 0 \le t \le 2\pi/\omega, \\ 0, & t > 2\pi/\omega. \end{cases}$$

(b) Using your answer to (a) (with $A = 1$ V), calculate $I(2\pi/\omega)$.

Solution. Using (1.5), we have

$$I(2\pi/\omega) = \frac{1 - e^{-2\pi/\omega}}{2(\omega^2+1)}.$$

(c) Using your answer to (b) as an “initial” condition, solve the differential equation for $t > 2\pi/\omega$.

Solution. When the voltage has been turned off, the resulting equation is

$$\dot{I} + I = 0,$$

the solution of which is trivially $I = ke^{-t}$. Solving for k, we have

$$\begin{aligned} I(2\pi/\omega) &= ke^{-2\pi/\omega} \\ k &= I(2\pi/\omega)e^{2\pi/\omega} \\ I(t) &= \frac{1 - e^{-2\pi/\omega}}{2(\omega^2+1)} e^{-(t-2\pi/\omega)}, \quad t > 2\pi/\omega. \end{aligned}$$