MATH 351-010 — Engineering Mathematics I
Prof. D. A. Edwards — Due: Nov. 18, 2022

Homework Set 9

Read sections P4.1, P4.3.

Section P4.1

1. Consider the following matrix and vectors:

$$C_1 = \begin{pmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{pmatrix}, \qquad \mathbf{z}_1 = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}, \quad \mathbf{z}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{z}_3 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

   By direct multiplication, show that the listed vectors $\mathbf{z}_i$ are eigenvectors for C_1, and find the corresponding eigenvalues.

2. Let λ_j be the jth eigenvalue of A, $\mathbf{z}_j$ be the eigenvector of A corresponding to λ_j. In addition, let α be a real number and β_j be the jth eigenvalue of B. In each part below, show that $\mathbf{z}_j$ is also an eigenvector for B corresponding to β_j and prove the listed identity.
   (a) If $B = \alpha A$, show that $\beta_j = \alpha\lambda_j$.
   (b) If $B = A + \alpha I$, show that $\beta_j = \lambda_j + \alpha$.
   (c) If $B = p(A)$, where $p(x)$ is a polynomial in x, show that $\beta_j = p(\lambda_j)$.

3. Consider the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}.$$

   (a) Find the eigenvalues of A.
   (b) Find the eigenvectors of A.

4. Repeat the steps of #3 for the matrix

$$A^T = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}.$$

   Discuss any similarities and differences.

Section P4.3

5. Let $A \in \mathcal{R}^{n\times n}$, $\lambda \neq 0$ an eigenvalue for A. Show that if $\mathbf{x}$ an eigenvector for λ, $\mathbf{x} \in \operatorname{col} A$.

6. Let A be a matrix whose columns all add up to the same constant δ. Show that δ is an eigenvalue of A.

7. Find the eigenvalues and corresponding eigenspaces for the following matrices:

$$A_1 = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}. \tag{9.1}$$

8. Consider the following matrix:

$$C = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(Matrices of this form occur regularly in physical problems. For instance, this matrix could represent the diffusion of heat in a one-dimensional bar.) Find the eigenvalues and a basis for each of the corresponding eigenspaces.

9. Find the eigenvalues and eigenvectors of the matrix

$$D = \begin{pmatrix} \alpha & -1 \\ \alpha - 1 & 0 \end{pmatrix}.$$

10. The *Cayley-Hamilton theorem* states that every matrix $A \in \mathcal{R}^{n\times n}$ is a root of its characteristic polynomial. So if

$$p_A(\lambda) = \sum_{j=0}^{n} a_j \lambda^j,$$

then

$$p_A(A) = \sum_{j=0}^{n} a_j A^j = O.$$

(Note that $A^0 = I$.) Verify that the matrix A_1 in (9.1) satisfies its characteristic polynomial.