MATH 351-010 Engineering Mathematics I
Prof. D. A. Edwards Due: Sept. 30, 2022

Homework Set 4 (Revised)

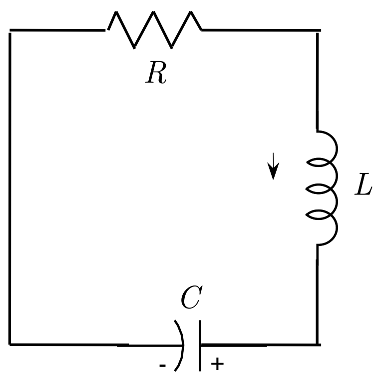
Read sections Z3.4, Z3.8.

Section Z3.8(a)

1. A mass weighing 96 pounds stretches a spring 8 feet.
   (a) Determine the amplitude and period of motion if the mass is initially released from a point 2 feet below the equilibrium position with a downward velocity of 1 ft/s.
   (b) How many complete cycles will the mass have made at the end of 4π seconds?

2. After a mass weighing 25 pounds is attached to a 8-foot spring, the spring measures 13 feet. This mass is removed and replaced with another mass that weighs 16 pounds. The entire system is placed in a medium that offers a damping force numerically equal to the instantaneous velocity.
   (a) Find the equation of motion if the mass is initially released from a point 1 foot below the equilibrium position with an upward velocity of -4 ft/s.
   (b) Write your solution in the amplitude-phase form given in class.

3. A mass weighing 36 pounds stretches a spring 2 feet. The mass is then attached to a dashpot that damps the motion with a force equal to β times the velocity, where β is a positive constant.
   (a) Write the equation of motion for the displacement $x(t)$.
   (b) Find the values of β such that the motion is overdamped, underdamped, and critically damped.



4. Consider the *series RLC* circuit shown in the figure above. There is an initial voltage on the capacitor of V_0.

   (a) Use the fact that the sum of the voltages around the loop must be zero to obtain the ODE
$$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0. \tag{4.1a}$$

   Initially, the current is I_0.

   (b) Show that the resulting initial conditions are
$$I(0) = I_0, \qquad \dot{I}(0) = -\frac{V_0 + RI_0}{L}. \tag{4.1b}$$

   (c) Under what conditions will the circuit be underdamped? overdamped? critically damped?

   (d) Solve the system when $L = 1$, $C = 1/5$, $I_0 = 2$, $R = V_0 = 4$.

5. Consider two species in a closed environment: a predator (population f) and its prey (population h). An early model for the evolution of the populations is, after suitable nondimensionalization,

$$\dot{h} = h(1-f), \tag{4.2a}$$
$$\dot{f} = \alpha^2 f(h-1). \tag{4.2b}$$

One *fixed point* of this equation (where $\dot{h} = \dot{f} = 0$) is $(h, f) = (0, 0)$. (Clearly if there are no specimens to begin with, the size of the populations will not change.)

(a) Find the other, more realistic fixed point (h_*, f_*).

To examine what happens in the neighborhood of the populations near the fixed point in (a), we let

$$\epsilon x(t) = h(t) - h_*, \quad \epsilon y(t) = f(t) - f_*; \qquad 0 < \epsilon \ll 1. \tag{4.3}$$

(b) Show that if we substitute (4.3) into (4.2) and take the limit that $\epsilon \to 0$, the resulting system for x and y is

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= \alpha^2 x. \end{aligned} \tag{4.4}$$

(c) Reduce the system (4.4) to a single second-order ODE for $x(t)$.

(d) Show that the solution to the system (4.4) is given by

$$\begin{aligned} x(t) &= c_1 \cos \alpha t + c_2 \sin \alpha t, \\ y(t) &= \alpha(c_1 \sin \alpha t - c_2 \cos \alpha t). \end{aligned}$$

Section Z3.4

(For the problems in this section, use the method of undetermined coefficients.)

6. Find the general solution to the differential equation

$$\ddot{y} + y = \sin \omega t.$$

Be sure to account for all ω.

7. Find the solution to the system

$$\ddot{y} + 16y = 32t^2, \qquad y(0) = 0, \quad \dot{y}(0) = 12.$$

8. Find the general solution to the differential equation

$$\ddot{y} + 3\dot{y} + 2y = 12e^{2t} + 18e^t.$$

9. Find the general solution to the differential equation

$$\ddot{y} - 9y = 6e^{3t}.$$

10. Consider the equation

$$\ddot{y} + 3\dot{y} = 10e^{-2t}\sin t, \qquad y(0) = 3, \qquad \dot{y}(0) = -2.$$

(a) Find the solution.
(b) Show that $y(\pi) \approx 3$.