MATH 351-010 Engineering Mathematics I
Prof. D. A. Edwards Due: Sept. 23, 2022

Homework Set 3

Read sections Z3.1–Z3.3.

Section Z3.1(b)

1. For the equation
$$\ddot{y} + 3\dot{y} - 10y = 0,$$
find the fundamental set $\{y_1(t), y_2(t)\}$ where
$$y_1(0) = 1, \quad \dot{y}_1(0) = 0; \qquad y_2(0) = 0, \quad \dot{y}_2(0) = 1.$$

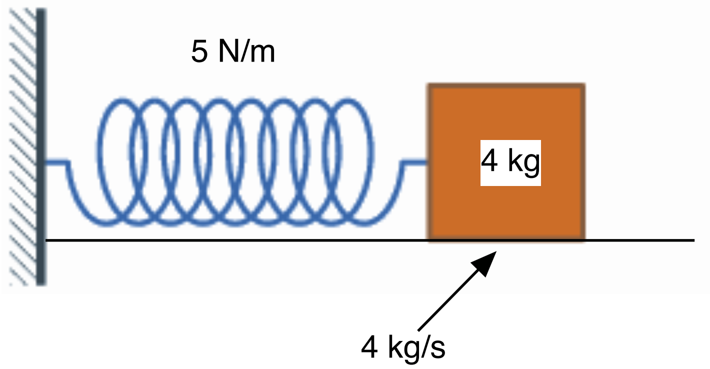
2. Consider the equation
$$(\sin t)\ddot{y} + t\dot{y} + \frac{3y}{t^2 + 2t - 3} = 0.$$
Find all intervals where this equation is guaranteed to have a unique solution.

3. Consider the ODE
$$y^{(3)} - 7\dot{y} + 6y = 0. \tag{3.1}$$
   (a) By direct substitution, show that three solutions of (3.1) are given by $\{e^t, e^{2t}, e^{-3t}\}$.
   (b) Show that the Wronskian of these three solutions is constant.

Sections Z3.2, 3.3

You may use complex arithmetic as much as you wish to simplify the algebra, but all final expressions should be expressed as real functions.

4. Consider the equation
$$3\ddot{y} + 7\dot{y} + 2y = 0.$$
   (a) Find the general solution.
   (b) Describe the behavior of the solution as $t \to \infty$. Is the solution overdamped, underdamped, critically damped, or none of those?



5. A spring with stiffness $k = 5$ N/m is damped with damping constant $b = 4$ kg/s and is attached to a weight with mass $M = 4$ kg (see figure).
   (a) Show that the resulting equation for the displacement x is

$$4\ddot{x} + 4\dot{x} + 5x = 0. \tag{3.2}$$

   The initial displacement is 2 m, and the initial velocity is -2 m/s.
   (b) Solve (3.2) subject to these conditions.
   (c) Equipment in the lab can measure the displacement down to a level of 1 mm. Estimate the time t_* after which the displacement will *always* remain below the threshold level. (*Hint: Use the amplitude-phase form.*)

6. Prove the following statements:
   (a)
$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \qquad \Longrightarrow \qquad \dot{y}(0) = \alpha c_1 + \beta c_2.$$
   (b)
$$y(t) = e^{\alpha t}(c_1 t + c_2) \qquad \Longrightarrow \qquad \dot{y}(0) = c_1 + \alpha c_2.$$

7. Consider the equation

$$\ddot{x} + 2\dot{x} + 2x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = \alpha \geq 0.$$

   (a) Construct the solution $x(t)$.
   (b) Show that $x(t_*) = 0$ whenever

$$\tan t_* = -\frac{1}{\alpha + 1}. \tag{3.3}$$

   (c) By considering the limits $\alpha \to 0$ and $\alpha \to \infty$, construct upper and lower bounds on the smallest positive t_*.

8. If the roots of $a\lambda^2 + b\lambda + c = 0$ are λ_1 and λ_2, then from notes in class we know that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are solutions of the ODE $a\ddot{y} + b\dot{y} + cy = 0$.
   (a) Show that
   $$\phi(t; \lambda_1, \lambda_2) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \quad \lambda_1 \neq \lambda_2,$$
   is also a solution of the equation.
   (b) Calculate
   $$\lim_{\lambda_2 \to \lambda_1} \phi(t; \lambda_1, \lambda_2)$$
   and verify that the solution thus obtained is the second solution in the case of a repeated root.

9. The displacement $x(t)$ of a spring is governed by the following equation:
   $$\ddot{x} + 4\dot{x} + 4x = 0, \qquad x(0) = x_0 > 0, \quad \dot{x}(0) = v_0.$$
   (a) Construct the solution to this problem.
   (b) Show that $x(t_*) = 0$ if and only if $v_0/x_0 < -2$. In this case, how many times does the solution cross the t-axis? Interpret your results in terms of initial velocity and displacement.

10. Find the solution of
   $$y^{(3)} - 3\ddot{y} + 4y = 0, \qquad y(0) = 4, \quad \dot{y}(0) = 4, \quad \ddot{y}(0) = 9.$$