Updates

1. Exam III will be administered on Wednesday, Nov. 30. It will cover up through the end of Poole, chapter 4, so you should attempt the first two problems on this homework before then.
2. Office hours on Thursday, Dec. 1 will be held from 11–12.
3. This homework is due on the last day of classes, Wednesday, Dec. 7.

Homework Set 10 (Revised)

Read sections P4.4, Z2.9, Z10.1, Z10.2, Z11.2.

Section P4.4

1. Consider the following matrix:

$$F_2 = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -2 & -2 \\ 6 & 6 & 6 \end{pmatrix}.$$

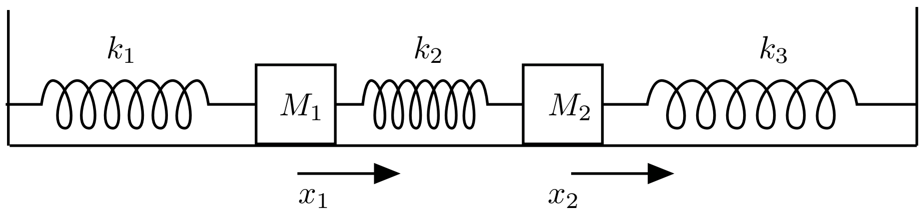
We wish to calculate the eigenvalues of this matrix *without* using the characteristic polynomial.

(a) Use facts about determinants to explain why $\lambda_1 = 0$.
(b) Use your answer to Homework Set 9, #6 to determine another eigenvalue.
(c) Use facts about the trace to determine the third eigenvalue.
(d) Calculate the matrices S, Λ, and S^{-1} needed to diagonalize F_2.

2. Consider the following matrix and vector:

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$

(a) Find the eigenvalues of A.
(b) Find the corresponding eigenvectors of A.
(c) Write the spectral decomposition $A = S\Lambda S^{-1}$.
(d) Calculate $[A\mathbf{x}]_Z$ and $\Lambda[\mathbf{x}]_Z$. Verify that they are equal.
(e) Use your answer to part (c) to calculate $A^{10}\mathbf{x}$.



Section Z2.9

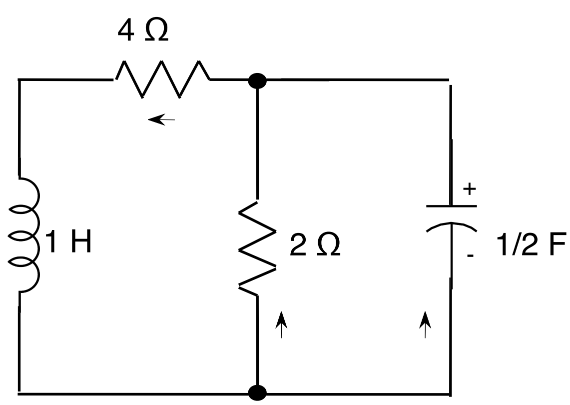
3. Consider the diagram above, which shows two bobs (mass M_j) connected by three springs to each other and external walls. Here x_j is the position of bob j measured from its equilibrium position (not necessarily indicated in the diagram).

   (a) Show that the system governing the motion of each bob is given by

$$M_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1), \tag{10.1a}$$
$$M_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1). \tag{10.1b}$$

   Be sure to clearly explain the sign of each term.

   (b) By introducing the velocity v_j of each bob, write (10.1) as a system of four coupled *first-order* ODEs.



4. Consider the circuit shown above.

The current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$\dot{I} = -4I - V, \tag{10.2a}$$
$$\dot{V} = 2I - V. \tag{10.2b}$$

   (b) Combine (10.2) to form a single second-order differential equation for V.
   (c) Solve (10.2) for V, then substitute into (10.2b) to find the solution for I.
   (d) Find the solution for V and I if $V(0) = 0$, $I(0) = 3$.

Section Z10.1

5. Consider the vectors

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 7 \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}.$$

   (a) Calculate the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.
   (b) Where are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?
   (c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?
   (d) Show (by deriving the coefficients of A, **NOT** by direct substitution) that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are the solutions of

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad A = \frac{1}{t-14}\begin{pmatrix} 15 & -2(t+1) \\ 7 & -t \end{pmatrix},$$

   and hence verify your answer to (c).

Section Z10.2.1/11.2(a)

6. Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} -4 & 4 & -2 \\ -1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix} \mathbf{x}.$$

   (a) Show that the eigenvalues for this matrix are $\lambda_1 = 0$, $\lambda_2 = -2$, and $\lambda_3 = -1$.
   (b) Find the general solution $\mathbf{x}(t)$ of this system.
   (c) What happens to the solution as $t \to \infty$?

7. Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \mathbf{x}. \tag{10.3}$$

   (a) Show that the eigenvectors for this system are $\mathbf{z}_1 = (1,2)^T$, $\mathbf{z}_2 = (1,-1)^T$.
   (b) Find the general solution $\mathbf{x}(t)$ of this system.
   (c) Sketch the phase plane for this system. Classify the fixed point.
   (d) Find the solution of the initial-value problem given by (10.3) and $\mathbf{x}(0) = (0,3)^T$.

Section Z10.2.3/11.2(b)

8. Consider the matrix

$$A = \begin{pmatrix} 2 & \gamma \\ 2 & 2 \end{pmatrix} \mathbf{x}.$$

   (a) Find the eigenvalues and eigenvectors of A.
   (b) Why will the qualitative nature of the phase plane change when γ passes through the critical value zero?
   (c) Sketch phase planes for γ less than and greater than zero. Classify the fixed point for $\gamma < 0$.
   (d) Find the general solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$. (*Hint: You may leave your solution in the complex form if needed.*)

9. We reconsider the circuit in #4.
   (a) Write the system (10.2) in matrix-vector form.
   (b) Solve the system for I and V using eigenvalues and eigenvectors. Confirm that your answer agrees with #4(c).
   (c) Does the solution depend on the initial data as $t \to \infty$?
   (d) Find the solution for V and I if $V(0) = 0$, $I(0) = 3$. Confirm that your answer agrees with #4(d).