Beats

In class we considered the solution of the system

$$\ddot{u} + k^2 u = F_0 \cos \omega t, \qquad u(0) = 0, \quad \dot{u}(0) = 0,$$

and found the answer to be

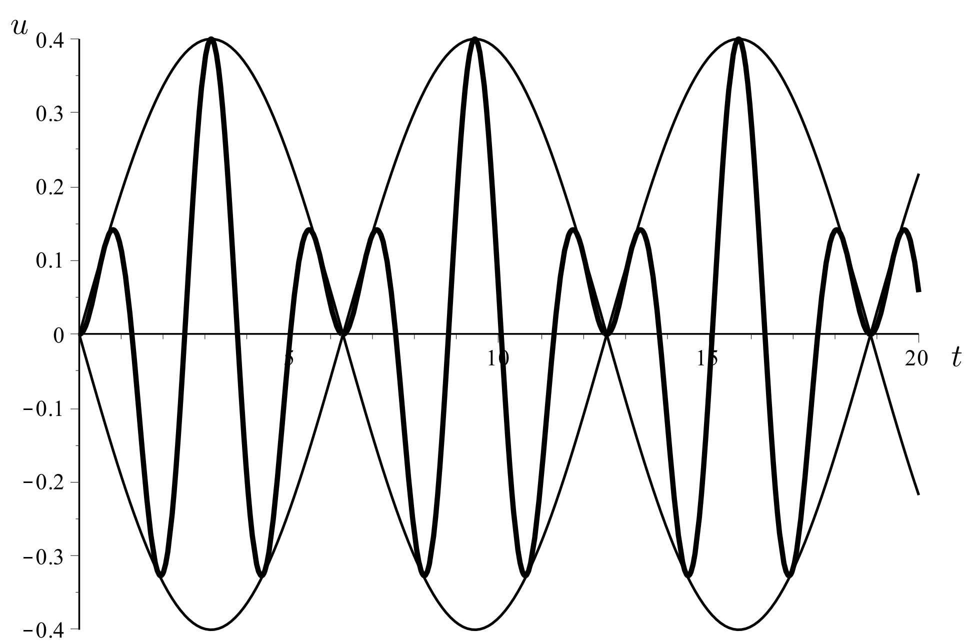
$$u = \frac{2F_0}{k^2 - \omega^2} \sin\left(\frac{(k-\omega)t}{2}\right) \sin\left(\frac{(k+\omega)t}{2}\right), \qquad k \neq \omega.$$

We consider the first two factors to be a time-varying amplitude (or envelope).

If $k = \omega$, the envelope grows linearly and we have

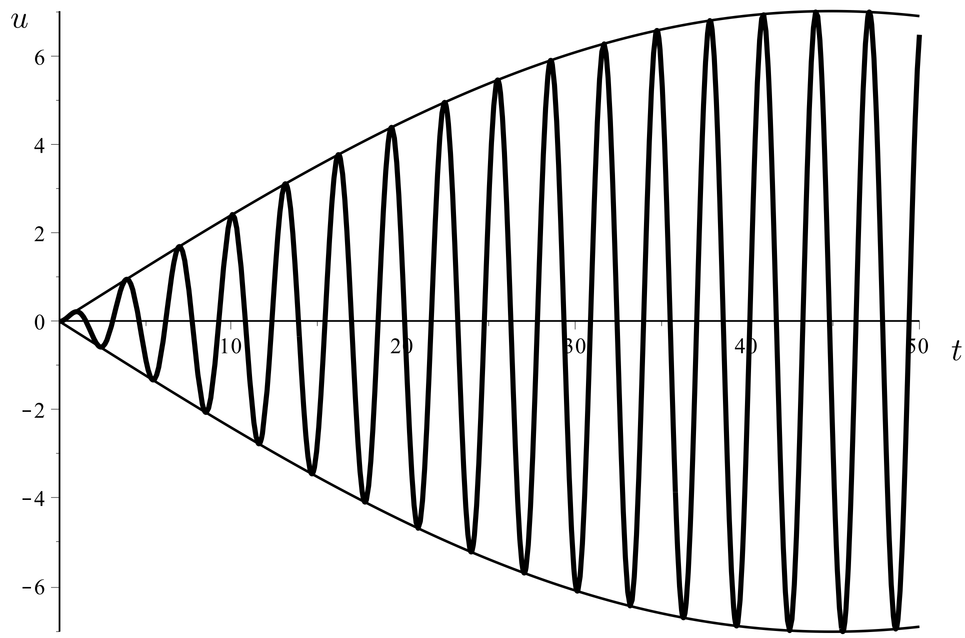
$$u = \frac{F_0 t}{2k} \sin kt.$$

Now take $k = 2$ and $F_0 = 1$. Here are some graphs of the solution for various values of ω.



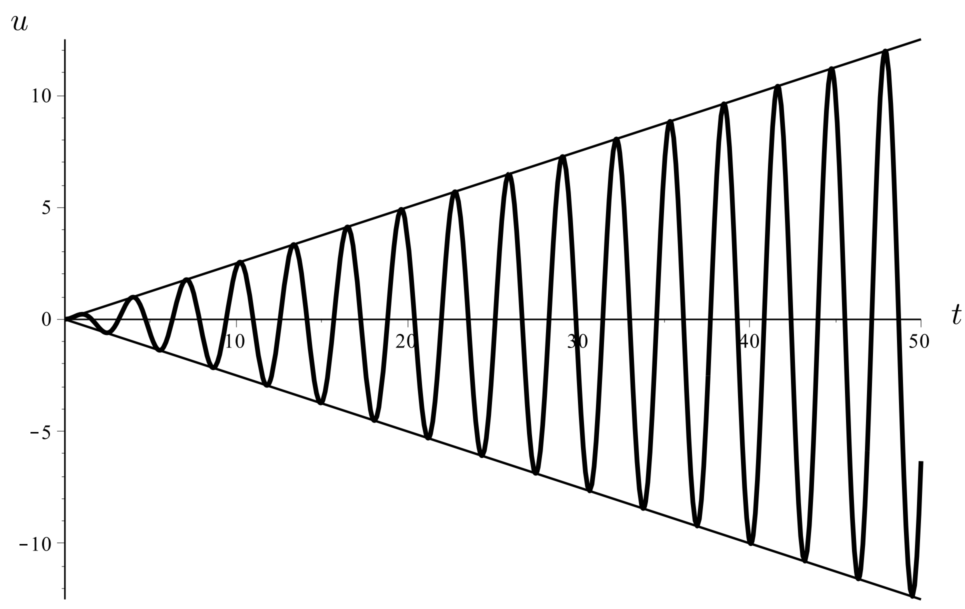
$u(t)$ *vs.* t for $\omega = 3$. Thin curve: envelope. Thick curve: solution.

In this case the forcing frequency is far from the natural frequency $\lambda = 2$, so we have languid oscillations within the beats.



$u(t)$ *vs.* t for $\omega = 2.07$. Thin curve: envelope. Thick curve: solution.

As the forcing frequency nears the natural frequency $k = 2$, we see much more rapid oscillations within the beats, but note that the envelope eventually reaches a maximum and will return to zero.



$u(t)$ *vs.* t for $\omega = 2$. Thin line: envelope. Thick curve: solution.

If the forcing frequency is the same as the natural frequency, the envelope increases linearly and so do the amplitude of the oscillations.