Proof Techniques

In this class, you will be asked to prove facts about vectors and matrices. Providing an example is not enough; you must prove that the statement is always true. There are three major techniques that you will use.

Direct Proof

The direct proof technique is the most straightforward. Using all the facts given in the problem, prove the hypothesis directly.

Example. Let x be a 3-digit number. Prove that x is divisible by 3 if and only if the sum of its digits is.

Proof. Since x is a 3-digit number, we may write $x = 100h + 10t + u$, where h is the digit in the hundreds place, t is the digit in the tens place, and u is the digit in the ones place. Hence, we need to check if $x/3$ is an integer. But

$$\frac{x}{3} = \frac{100h + 10t + u}{3} = 33h + 3t + \frac{h + t + u}{3}.$$

Since $33h + 3t$ is obviously an integer, $x/3$ is an integer if and only if the last fraction is an integer, which is what we wished to prove.

Proof by Contradiction

In this case, we assume that the conclusion is false, and work until we arrive at a contradiction. Hence the conclusion must be true.

Example. Let q be an integer. Show that if $q \neq 2$, the set $\{q, q + 1\}$ contains at most one prime.

Proof. Suppose that q and $q + 1$ are both prime. (This is the false conclusion.) Since the only even prime is 2 and $q \neq 2$, q must be odd, since it was assumed prime. Hence let $q = 2m + 1$ for some positive integer m. But then $q + 1 = 2m + 2 = 2(m + 1)$, so $q + 1$ is even and not 2. So $q + 1$ is not a prime, which is a contradiction.

Proof by Induction

These proofs are used when we are proving something that involves an integer n (usually the dimension of a vector or matrix). First, prove the result for a specific low value of n (usually 1 or 2). Then assume the statement is true for n, then prove that implies the statement is true for $n+1$. This completes the proof, because if it is true for $n+1$, that implies it is true for $(n+1)+1$, etc.

Example. Show that the sum of the first n integers is $n(n+1)/2$.

Proof. First we prove for $n=1$ by summing the first integer:

$$1 = \frac{1(1+1)}{2} = \frac{n(n+1)}{2}.$$

Assume the statement is true for n. Then summing the first $n+1$ integers, we have

$$\sum_{i=1}^{n+1} i = (n+1) + \sum_{i=1}^{n} i = n+1+\frac{n(n+1)}{2} = \frac{2(n+1)+n^2+n}{2} = \frac{(n+1)(n+2)}{2}$$
$$= \frac{(n+1)[(n+1)+1]}{2},$$

so the statement is true for $n+1$.