# Supplemental Study Material Solutions

1. (BH) Calculate the matrix $A$ which has the following properties:

$$\lambda_1(A) = 2, \qquad \lambda_2(A) = -5, \qquad \mathbf{z}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \mathbf{z}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

*Solution.* Using the given statements, we have that

$$\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -5 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \qquad \Longrightarrow \qquad S^{-1} = \frac{1}{1}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}.$$

Therefore, we have that

$$A = S\Lambda S^{-1} = S\begin{pmatrix} 2 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 14 & -4 \\ 15 & -5 \end{pmatrix} = \begin{pmatrix} 44 & -14 \\ 147 & -47 \end{pmatrix}.$$

2. Consider the following matrix and vector:

$$A = \begin{pmatrix} -5 & -3 \\ 6 & 4 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

(a) (BH) Find the eigenvalues of $A$.

*Solution.* Calculating the characteristic polynomial, we have

$$\det(A - \lambda I) = \begin{vmatrix} -5-\lambda & -3 \\ 6 & 4-\lambda \end{vmatrix} = -20 + \lambda + \lambda^2 + 18 = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2).$$

Setting the characteristic polynomial equal to zero, we have that $\lambda_1 = 1$ and $\lambda_2 = -2$.

(b) (BH) Find the corresponding eigenvectors of $A$.

*Solution.* Solving for the eigenspaces, we obtain

$$(A - \lambda_1 I)\mathbf{z}_1 = \begin{pmatrix} -6 & -3 \\ 6 & 3 \end{pmatrix}\mathbf{z}_1 = \mathbf{0}.$$

The rows are multiples of one another, so we see that $6x + 3y = 0$, so a typical eigenvector is $(1, -2)^T$. Similarly, we have

$$(A - \lambda_2 I)\mathbf{z}_2 = \begin{pmatrix} -3 & -3 \\ 6 & 6 \end{pmatrix}\mathbf{z}_2 = \mathbf{0}.$$

The rows are multiples of one another, so we see that $6x+6y=0$, so a typical eigenvector is $(-1,1)^T$.

(c) (BH) Write the spectral decomposition $A = S\Lambda S^{-1}$.

*Solution.* **IMPORTANT: The answer to this problem is dependent on the ordering of the eigenvalues and the choice of eigenvectors**. Using our answers from (a) and (b), we have

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \quad \Longrightarrow \quad S^{-1} = -\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

(d) (BH) Calculate $[A\mathbf{x}]_Z$ and $\Lambda[\mathbf{x}]_Z$. Verify that they are equal.

*Solution.* To calculate the coordinates in the $Z$ basis, we need the transition matrix, which is simply the inverse of the matrix of eigenvectors. Thus the transition matrix is $S^{-1}$. So we have

$$[A\mathbf{x}]_Z = S^{-1}\begin{pmatrix} -5 & -3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix} = -\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ -10 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix},$$

$$\Lambda[\mathbf{x}]_Z = \Lambda\left[-\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

(e) (BH) Use your answer to part (c) to calculate $A^{10}\mathbf{x}$.

*Solution.*

$$A^{10}\mathbf{x} = (S\Lambda S^{-1})^{10}\mathbf{x} = S\Lambda^{10}S^{-1}\mathbf{x} = S\Lambda^{10}[\mathbf{x}]_Z = S\left[\begin{pmatrix} 1^{10} & 0 \\ 0 & (-2)^{10} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2^{11} \end{pmatrix} = \begin{pmatrix} 3-2^{11} \\ -6+2^{11} \end{pmatrix} = \begin{pmatrix} -2045 \\ 2042 \end{pmatrix}.$$

(f) (MP) Calculate $A^{10}\mathbf{x}$ and check your answer with (e).

3. Let $A$ and $B$ be similar.

(a) (BH) Show that if $A$ is nonsingular, then $B$ is nonsingular.

*Solution.* Let $B = PAP^{-1}$. Then since $P$ is invertible, we have

$$\begin{aligned} BP &= PA \\ BPA^{-1} &= P \\ BPA^{-1}P^{-1} &= I. \end{aligned}$$

Thus $B^{-1} = PA^{-1}P^{-1}$, and hence $B$ is invertible.

(b) (BH) If $A$ is nonsingular, then $A^{-1}$ and $B^{-1}$ are similar.

*Solution.* By part (a), we have that $B^{-1} = PA^{-1}P^{-1}$, and hence $A^{-1}$ and $B^{-1}$ are similar.

Now consider the particular matrices

$$A = \begin{pmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

(Do **NOT** use these matrices for parts (a) and (b); those proofs should be for *any* $A$ and $B$.)

(c) (MP) Verify that $A$ and $B$ are similar and invertible, and compute the matrix $P$ such that $A = PBP^{-1}$.

(d) (MP) Verify that $A^{-1}$ and $B^{-1}$ are similar, and compute the matrix $Q$ such that $A^{-1} = QB^{-1}Q^{-1}$.

(e) (BH) Show that it is possible to choose $P = Q$. Is that true of the matrices in Mathematica? What does that tell you about the uniqueness of $P$ and $Q$?

*Solution.* Taking the inverse of both sides of the $P$ equation, we have

$$A^{-1} = (P^{-1})^{-1}B^{-1}(P)^{-1} = PB^{-1}P^{-1},$$

which is exactly the same as the $Q$ equation with $Q$ replaced by $P$. However, $P$ and $Q$ are not the same in our Mathematica calculation, and hence they most not be unique.

In[ ]:= 
```
Quit[]
```

In[ ]:= 
```
$PrePrint = If[MatrixQ[#] || VectorQ[#], MatrixForm[#], #] &;
```

# HW1 (Checked)

# HW2 (Checked)

# HW3 (Checked)

# HW4 (Checked)

# HW5 (Checked)

# HW6 (Checked)

# HW7 (Checked)

# HW8 (Checked)

# HW9 (Checked)

# HW10 (Checked)

# SSM (Checked)

## Check

## Number 2f.

Calculate $A^{10}\boldsymbol{x}$, where

$A = \begin{pmatrix} -5 & -3 \\ 6 & 4 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

In[ ]:=
```
a2f = {{-5, -3}, {6, 4}}
x2f = {1, -4}
MatrixPower[a2f, 10].x2f
```

Out[ ]=
$$\begin{pmatrix} -5 & -3 \\ 6 & 4 \end{pmatrix}$$

Out[ ]=
$$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Out[ ]=
$$\begin{pmatrix} -2045 \\ 2042 \end{pmatrix}$$

## Number 3c.

Show that the matrices $A$ and $B$ are invertible, where

$A = \begin{pmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$, and find the matrix $P$ such that $A = P\,B\,P^{-1}$.

In[ ]:=

In[ ]:=
```
A = {{4, 4, 2, 3, -2}, {0, 1, -2, -2, 2},
   {6, 12, 11, 2, -4}, {9, 20, 10, 10, -6},
   {15, 28, 14, 5, -3}}
B = {{3, 0, 0, 0, 0}, {0, 5, 0, 0, 0},
   {0, 0, 5, 0, 0}, {0, 0, 0, 7, 0}, {0, 0, 0, 0, 3}}
```

Out[ ]=
$$\begin{pmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{pmatrix}$$

Out[ ]=
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

First we check that $A$ and $B$ are invertible:

```
In[]:= ainv = Inverse[A]
       binv = Inverse[B]
```

Out[]=

$$\begin{pmatrix} \frac{12}{35} & -\frac{4}{35} & -\frac{2}{35} & -\frac{17}{105} & \frac{2}{21} \\ 0 & \frac{7}{15} & \frac{2}{15} & \frac{2}{15} & -\frac{2}{15} \\ -\frac{2}{5} & -\frac{4}{5} & -\frac{1}{5} & -\frac{2}{15} & \frac{4}{15} \\ -\frac{13}{35} & -\frac{92}{105} & -\frac{46}{105} & -\frac{2}{105} & \frac{2}{7} \\ -\frac{27}{35} & -\frac{148}{105} & -\frac{74}{105} & -\frac{23}{105} & \frac{13}{21} \end{pmatrix}$$

Out[]=

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Since $B$ is diagonal, we expect it to be the matrix of eigenvalues of $A$ , which it is:

```
In[]:= esys = Eigensystem[A]
```

Out[]=

```
{{7, 5, 5, 3, 3}, {{1, 0, 0, 3, 3}, {-1, 1, 0, -1, 1},
  {0, -1, 2, 0, 0}, {-2, 1, 1, 0, 2}, {4, -3, 1, 2, 0}}}
```

Then in order to construct the matrix $S$ of eigenvectors that has the columns in the right order, we have to pick the vectors out manually

```
In[]:= {esys[[2, 4]], esys[[2, 2]], esys[[2, 3]], esys[[2, 1]], esys[[2, 5]]}
```

Out[]=

$$\begin{pmatrix} -2 & 1 & 1 & 0 & 2 \\ -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 3 \\ 4 & -3 & 1 & 2 & 0 \end{pmatrix}$$

Note that the vectors have been slotted into rows, so the matrix $S$ is the transpose of this:

```
In[]:= smat = Transpose[%]
```

Out[]=

$$\begin{pmatrix} -2 & -1 & 0 & 1 & 4 \\ 1 & 1 & -1 & 0 & -3 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 & 0 \end{pmatrix}$$

This matrix is the matrix $P$ for which we are looking, which we can verify:

In[ ]:=
```
smat.B.Inverse[smat]
A
```

Out[ ]=

$$\begin{pmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{pmatrix}$$

Out[ ]=

$$\begin{pmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{pmatrix}$$

## Number 3d.

Compute the matrix $Q$ such that $A^{-1} = Q\, B^{-1}\, Q$.

Since $B^{-1}$ is diagonal, we expect it to be the matrix of eigenvalues of $A^{-1}$, which it is:

In[ ]:=
```
eisys = Eigensystem[ainv]
```

Out[ ]=

$$\left\{\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}\right\}, \left\{\left\{-1, \frac{1}{2}, \frac{1}{2}, 0, 1\right\}, \left\{2, -\frac{3}{2}, \frac{1}{2}, 1, 0\right\}, \{-1, 1, 0, -1, 1\}, \left\{0, -\frac{1}{2}, 1, 0, 0\right\}, \left\{\frac{1}{3}, 0, 0, 1, 1\right\}\right\}\right\}$$

Then in order to construct the matrix $S$ of eigenvectors that has the columns in the right order, we have to pick the vectors out manually

In[ ]:=
```
{eisys[[2, 2]], eisys[[2, 3]], eisys[[2, 4]], eisys[[2, 5]], eisys[[2, 1]]}
```

Out[ ]=

$$\begin{pmatrix} 2 & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ -1 & 1 & 0 & -1 & 1 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix}$$

Note that the vectors have been slotted into rows, so the matrix $Q$ is the transpose of this:

In[ ]:=
```
qmat = Transpose[%]
```

Out[ ]=

$$\begin{pmatrix} 2 & -1 & 0 & \frac{1}{3} & -1 \\ -\frac{3}{2} & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 & \frac{1}{2} \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

This matrix is the matrix $Q$ for which we are looking, which we can verify:

In[ ]:=
```
qmat.binv.Inverse[qmat]
ainv
```

Out[ ]=

$$\begin{pmatrix} \frac{12}{35} & -\frac{4}{35} & -\frac{2}{35} & -\frac{17}{105} & \frac{2}{21} \\ 0 & \frac{7}{15} & \frac{2}{15} & \frac{2}{15} & -\frac{2}{15} \\ -\frac{2}{5} & -\frac{4}{5} & -\frac{1}{5} & -\frac{2}{15} & \frac{4}{15} \\ -\frac{13}{35} & -\frac{92}{105} & -\frac{46}{105} & -\frac{2}{105} & \frac{2}{7} \\ -\frac{27}{35} & -\frac{148}{105} & -\frac{74}{105} & -\frac{23}{105} & \frac{13}{21} \end{pmatrix}$$

Out[ ]=

$$\begin{pmatrix} \frac{12}{35} & -\frac{4}{35} & -\frac{2}{35} & -\frac{17}{105} & \frac{2}{21} \\ 0 & \frac{7}{15} & \frac{2}{15} & \frac{2}{15} & -\frac{2}{15} \\ -\frac{2}{5} & -\frac{4}{5} & -\frac{1}{5} & -\frac{2}{15} & \frac{4}{15} \\ -\frac{13}{35} & -\frac{92}{105} & -\frac{46}{105} & -\frac{2}{105} & \frac{2}{7} \\ -\frac{27}{35} & -\frac{148}{105} & -\frac{74}{105} & -\frac{23}{105} & \frac{13}{21} \end{pmatrix}$$