Updates

1. Exam III will be administered on Friday, Nov. 22. You will need to bring a small blue book, as well as your laptop.
2. Exam III will cover material up through Homework Set 9.
3. The third honors assignment is due Monday, Dec. 2.

Homework Set 10 (12/7 Version)

Read sections 4.1–4.3.

Section 4.2

1. (BH) exercise 65
2. (BH) Use Cramer's Rule to solve

$$\begin{aligned} 4x + y &= -1, \\ -3x - 2y &= -3. \end{aligned}$$

Section 4.1

3. Consider the following matrices and vectors:

$$C_1 = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad \mathbf{z}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{z}_2 = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}, \quad \mathbf{z}_3 = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{pmatrix}, \quad \mathbf{z}_1 = \begin{pmatrix} 2 \\ 2 \\ -4 \\ 2 \end{pmatrix}, \quad \mathbf{z}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{z}_3 = \begin{pmatrix} 4 \\ 0 \\ -2 \\ 0 \end{pmatrix},$$

$$\mathbf{z}_4 = \begin{pmatrix} 28 \\ 28 \\ 36 \\ 5 \end{pmatrix}.$$

(a) (BH) By direct multiplication, show that the listed vectors $\mathbf{z}_i$ are eigenvectors for C_1, and find the corresponding eigenvalues.

(b) (MP) By direct multiplication, show that the listed vectors $\mathbf{z}_i$ are eigenvectors for C_2, and find the corresponding eigenvalues.

4. (BH) Consider the following matrix and eigenvalues:

$$C_3 = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \qquad \lambda_1 = 1, \quad \lambda_2 = 5.$$

Find the eigenvectors for each of the eigenvalues.

5. (BH) Let A be an invertible matrix, and $A\mathbf{z} = \lambda\mathbf{z}$. Show that λ^{-1} is an eigenvalue of A^{-1}, and find the corresponding eigenvector.

Section 4.3

6. Let $A, B \in \mathcal{R}^{n\times n}$.
   (a) (BH) Prove that the sum of all the eigenvalues of $A+B$ is the sum of all the eigenvalues of A and B individually.
   (b) (MP) Verify your answer to (a) with two matrices in $\mathcal{R}^{4\times 4}$ who entries are random numbers in $[-2, 2]$.

7. Let $A, B \in \mathcal{R}^{n\times n}$.
   (a) (BH) Prove that the product of all the eigenvalues of AB is the product of all the eigenvalues of A and B individually.
   (b) (MP) Verify your answer to (a) with two matrices in $\mathcal{R}^{4\times 4}$ who entries are random numbers in $[-2, 2]$.

8.
   (a) (BH) Find all the eigenvalues and associated eigenvectors of each of the following matrices.

$$A_{\mathrm{a}} = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix}, \qquad A_{\mathrm{b}} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 4 \\ 0 & -4 & 0 \end{pmatrix}.$$

   (b) (MP) Calculate the eigenvalues and associated eigenvectors for the following matrix:

$$\begin{pmatrix} 9 & -4 & -13 \\ 7 & -2 & -13 \\ 3 & -2 & -3 \end{pmatrix}.$$

   Be sure to simplify your answer.

9. (BH) Consider the following matrix:

$$F = \begin{pmatrix} -2 & -2 & 4 \\ 0 & 2 & -4 \\ 0 & 3 & -6 \end{pmatrix}.$$

We wish to calculate the eigenvalues of this matrix *without* using the characteristic polynomial.

(a) Use facts about determinants to explain why $\lambda_1 = 0$.
(b) Verify that $\mathbf{e}_1$ is an eigenvector to obtain λ_2.
(c) Use facts about the trace to determine the third eigenvalue.

10. (BH) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}. \tag{10.1}$$

(a) Find the eigenvalues of A and their algebraic multiplicities.

The *Cayley-Hamilton Theorem* says that if

$$p_A(\lambda) = a_0 + \sum_{j=1}^{n} a_j \lambda^j$$

is the characteristic polynomial of A, then

$$p_A(A) = a_0 I + \sum_{j=1}^{n} a_j A^j = O.$$

(b) Verify the Cayley-Hamilton Theorem for the matrix in (10.1).