Theory for Second- and Higher-Order ODEs

For the second-order homogeneous constant coefficient ODE

$$a\ddot{y} + b\dot{y} + cy = 0, \tag{1a}$$

we found that substituting in $y = e^{\lambda t}$ yields the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0,$$

which has two roots λ_1 and λ_2. The general solution $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ satisfies (1a) along with the initial conditions

$$y(0) = y_0, \qquad \dot{y}(0) = \dot{y}_0, \tag{1b}$$

with constants given by

$$c_1 = \frac{\dot{y}_0 - \lambda_2 y_0}{\lambda_1 - \lambda_2}, \qquad c_2 = \frac{\lambda_1 y_0 - \dot{y}_0}{\lambda_1 - \lambda_2}, \tag{2}$$

which exist as long as $\lambda_1 \neq \lambda_2$.

In general, for the linear equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = g(t), \tag{3a}$$

there is a unique solution satisfying

$$y(t_0) = y_0, \qquad \dot{y}(t_0) = \dot{y}_0 \tag{3b}$$

in any interval I containing t_0 where p, q, and g are all continuous. This is because at $t = t_0$ you would know all the terms except the first in (3a), which means you can solve for $\ddot{y}(t_0)$ to advance the solution forward.

If we make (3a) **HOMOGENEOUS** by setting $g(t) = 0$ and find two solutions y_1 and y_2, the general solution $y = c_1 y_1(t) + c_2 y_2(t)$ satisfies (3b) with constants

$$c_1 = \frac{y_0 \dot{y}_2(t_0) - \dot{y}_0 y_2(t_0)}{W(t_0)}, \qquad c_2 = \frac{-y_0 \dot{y}_1(t_0) + \dot{y}_0 y_1(t_0)}{W(t_0)}, \tag{4}$$

which exist as long as the *Wronskian*

$$W(t_0) = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}(t_0) \neq 0.$$

In that case, y_1 and y_2 are called *linearly independent* and form a *fundamental set.* [In the case of constant coefficients, (4) reduces to (2).]

The higher-order case is a direct extension of the above. The nth order linear equation

$$y^{(n)} + \sum_{j=0}^{n-1} p_j(t) y^{(j)} = g(t) \tag{5a}$$

can be written as n first-order equations, so we need n initial conditions

$$y^{(j)}(t_0). \tag{5b}$$

There is a unique solution of (5) in any interval I containing t_0 where the p_j and g are all continuous. This is because at $t = t_0$ you would know all the terms except the first in (5a), which means you can solve for $y^{(n)}(t_0)$ to advance the solution forward.

If we make (5a) **HOMOGENEOUS** by setting $g(t) = 0$ and find n solutions y_j, the general solution

$$y = \sum_{j=1}^{n} y_j(t)$$

can satisfy any set of initial conditions (5b) as long as the higher-order Wronskian

$$W(t_0) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \dot{y}_1 & \dot{y}_2 & \cdots & \dot{y}_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} (t_0) \neq 0.$$

In that case, the y_j are called *linearly independent* and form a *fundamental set.*