Sine and Cosine Series

We derived in class that the Fourier cosine series for the function

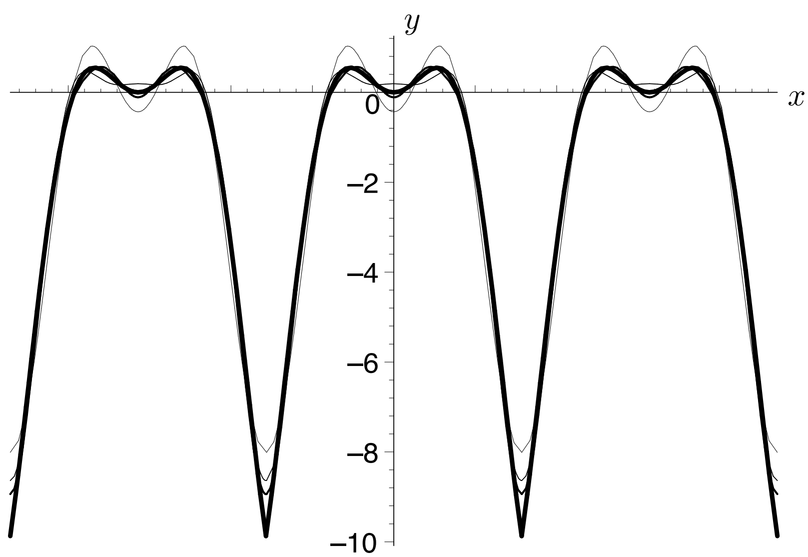
$$f(x) = x^2 \cos x, \qquad x \in (-\pi, \pi)$$

is given by

$$f(x) = -2 + \left(\frac{1}{2} + \frac{\pi^2}{3}\right)\cos x + 2\sum_{n=2}^{\infty}\left[\frac{(-1)^{n+1}}{(n+1)^2} + \frac{(-1)^{n-1}}{(n-1)^2}\right]\cos nx.$$

Note that $f(x)$ is smooth in its domain, but f' is discontinuous at $x = \pm\pi$.

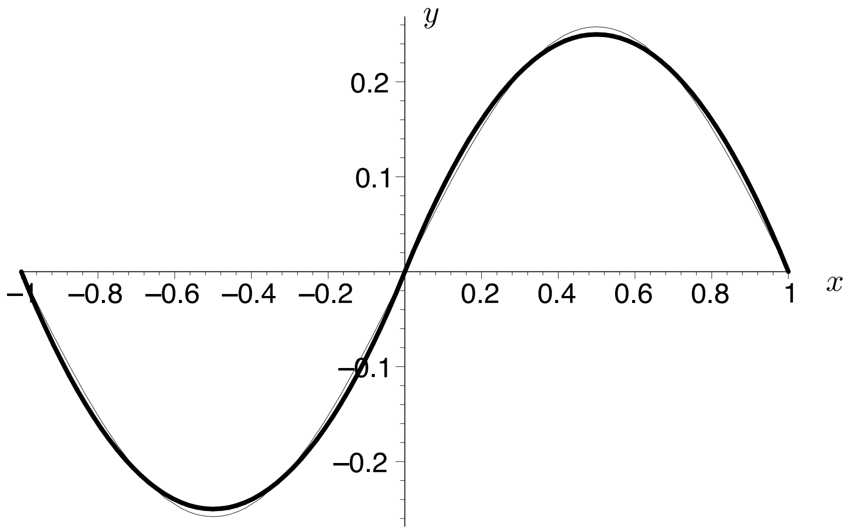
Below are plotted the extension of the function f to the domain $x \in (-3\pi, 3\pi)$ (thickest line), as well as the Fourier series taking the first three, four, and five terms in the sum.



In increasing order of thickness: Fourier series keeping the first three, four, and five terms in the series, as well as $f(x)$ *vs.* x.

The Fourier sine series f_{s} for the odd extension of the function $f(x) = x(1-x)$, $x \in [0,1]$ to the region $x \in [-1,1]$ is given by

$$f_{\mathrm{s}}(x) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin n\pi x.$$



In increasing order of thickness: f_{s} keeping the first one, two, and three nonzero terms in the series, as well as the odd extension of $f(x)$ *vs.* x.

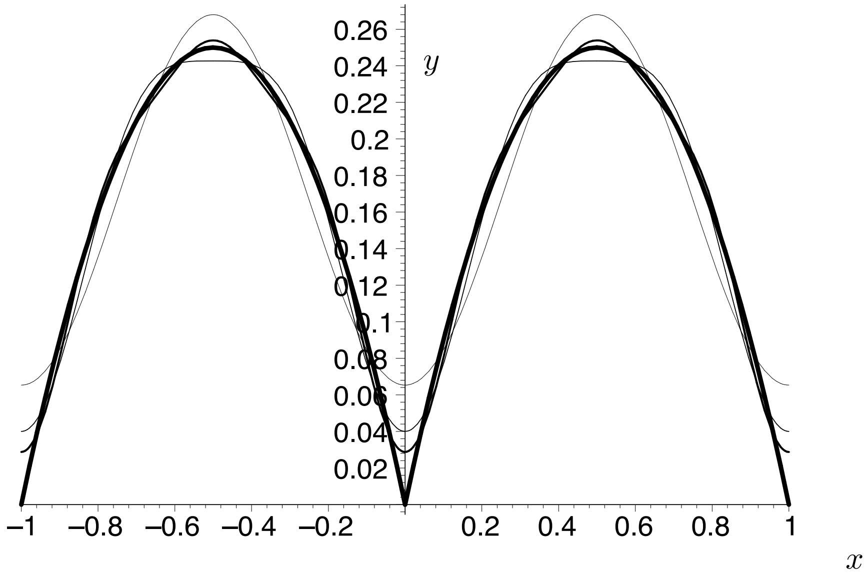
Remarks

1. The terms decay quickly (like n^{-3}).
2. The Fourier coefficient is zero if n is even.

The Fourier cosine series f_c for the even extension of the function $f(x) = x(1-x)$, $x \in [0, \pi]$ to the region $x \in [-\pi, \pi]$ is given by

$$f_c(x) = \frac{1}{6} - \frac{2}{\pi^2} \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{n^2} \cos n\pi x.$$

Below is plotted the even extension of the function f to the domain $x \in [-1, 1]$ (thickest line), as well as f_c taking the first several terms in the sum. Note that since the extension has a jump in f', we see that the terms decay slowly.



In increasing order of thickness: f_c keeping the first two, three, and four nonzero terms in the series, as well as the even extension of $f(x)$ *vs.* x.

Remarks

1. The terms do not decay as quickly (like n^{-2}). As a result, you need more of them to get the same accuracy as the sine series.
2. The series solutions are most inaccurate where the solution has a kink.
3. The Fourier coefficient is zero if n is odd.