# Analyzing Systems of First-Order ODEs

To analyze the system

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1}$$

the first step is always to find the eigenvalues $\lambda$, as they will allow you to characterize the fixed point at the origin:

| | |
|---|---|
| real $\lambda$, opposite signs | saddle point |
| two positive real $\lambda$ | unstable node |
| two negative real $\lambda$ | stable node |
| $\lambda = \pm i\beta$ | center |
| $\lambda = \alpha \pm i\beta$, $\alpha > 0$ | unstable spiral |
| $\lambda = \alpha \pm i\beta$, $\alpha < 0$ | stable spiral |

## The Phase Plane

If the eigenvalues are **COMPLEX**, determine the direction of flow by looking at what happens on the axes. For instance, on the $x_1$-axis where $x_2 = 0$, (1) becomes

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 \\ \dot{x}_2 &= a_{21}x_1 \end{aligned} \tag{2}$$

Use the sign of $\dot{x}_1$, $\dot{x}_2$ on the axes to determine the direction of flow. This and the classification of the fixed point will give you all you need to draw the phase plane.

If the eigenvalues are **REAL**, you need to find the two eigenvectors $\mathbf{z}_1$ and $\mathbf{z}_2$. Draw those, and use the sign of $\lambda$ to identify the direction of flow. Fill in trajectories in between the eigenvectors, remembering that the direction of flow varies smoothly.

## The General Solution

The general solution of (1) is given by

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t). \tag{3}$$

If the eigenvalues are **REAL**, you have already calculated the solutions, since in that case

$$\mathbf{x}^{(j)}(t) = e^{\lambda_j t}\mathbf{z}_j. \tag{4}$$

If the eigenvalues are **COMPLEX**, calculate the eigenvector $\mathbf{z}_+$ corresponding to $\lambda_+$. Then expand the complex solution into real and imaginary parts, which will provide the two solutions:

$$e^{\lambda_+ t}\mathbf{z}_+ = \underbrace{[\cdots]}_{\mathbf{x}^{(1)}(t)} + i\underbrace{[\cdots]}_{\mathbf{x}^{(2)}(t)}$$

If the eigenvalue is **REPEATED**, you can still use (4) for $\mathbf{x}^{(1)}$, and the second solution is given by

$$\mathbf{x}^{(j)}(t) = te^{\lambda t}\mathbf{z} + e^{\lambda t}\vec{\eta}, \qquad (A - \lambda I)\vec{\eta} = \mathbf{z}.$$

## Initial Conditions

If you need to solve initial conditions at $t = 0$, plug $t = 0$ into (3) and solve for $c_1$ and $c_2$.