Homework Set 9 Solutions

1. (BH) Consider the following problem:

$$\ddot{y} - (a+b)\dot{y} + aby = f(t), \qquad y(0) = 0, \quad \dot{y}(0) = 0, \quad a \neq b.$$

(a) Solve the problem using Laplace transforms.

Solution. Transforming the equation, we have

$$s^2\hat{y} - (a+b)s\hat{y} + ab\hat{y} = \hat{f}$$
$$\hat{y} = \frac{\hat{f}}{(s-a)(s-b)} = \hat{f}\left[\frac{1}{(s-a)(s-b)}\right],$$

where we have used the initial conditions. Remembering that we should focus on the partial-fraction decomposition of the bracketed quantity, we obtain

$$\frac{1}{(s-a)(s-b)} = \frac{c_1}{s-a} + \frac{c_2}{s-b}$$
$$1 = c_1(s-b) + c_2(s-a). \tag{A}$$

Substituting $s = b$ into (A) cancels the first term, so $c_2 = (b-a)^{-1}$. Substituting $s = a$ into (A) cancels the second term, so we have $c_1 = (a-b)^{-1}$. Hence we have

$$\hat{y} = \hat{f}\left[\frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)\right].$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)\right\} = \frac{e^{at} - e^{bt}}{a-b}, \tag{B}$$

using the convolution theorem we have

$$y(t) = \frac{1}{a-b}\int_0^t \left[e^{a(t-\tau)} - e^{b(t-\tau)}\right] f(\tau)\, d\tau.$$

(b) What is the Green's function for this problem?

Solution. The Green's function $g(t-\tau)$ is the function that multiplies $f(\tau)$ in the convolution. This is given by the expression in (B), so we have

$$g(t) = \frac{e^{at} - e^{bt}}{a-b},$$

where we have rewritten the right-hand side to reflect the new argument of g.

2. Consider the Laplace transform given by

$$\hat{f}(s) = \frac{1}{s^2(s^2+4)}.$$

(a) (BH) Evaluate the inverse Laplace transform by using a convolution involving the transform of the sine.

Solution. Since

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+2^2}, \qquad \mathcal{L}\left\{\frac{t}{2}\right\} = \frac{1}{2s^2},$$

we have

$$\begin{aligned}
\hat{f}(s) &= \left(\frac{1}{2s^2}\right)\left(\frac{2}{s^2+2^2}\right) \\
f(t) &= \frac{t}{2} * (\sin 2t) = \frac{1}{2}\int_0^t (t-\tau)\sin 2\tau\, d\tau = \frac{1}{2}\int_0^t u\, dv, \qquad u = t-\tau, \quad dv = \sin 2\tau\, d\tau \\
&= \frac{1}{2}\left\{\left[(t-\tau)\left(\frac{-\cos 2\tau}{2}\right)\right]_0^t - \int_0^t \frac{-\cos 2\tau}{2}(-d\tau)\right\} = \frac{t}{4} - \frac{1}{4}\left[\frac{\sin 2\tau}{2}\right]_0^t \\
&= \frac{t}{4} - \frac{\sin 2t}{8}.
\end{aligned}$$

(b) (MI) Evaluate the inverse Laplace transform by using a convolution involving the transform of the cosine. You must set up the integral by hand, but you can use Mathematica to actually integrate. Verify that your answer matches (a).

Solution. Since

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+2^2}, \qquad \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3},$$

we have

$$\begin{aligned}
\hat{f}(s) &= \left(\frac{1}{s^3}\right)\left(\frac{s}{s^2+2^2}\right) \\
f(t) &= \left(\frac{t^2}{2}\right) * (\cos 2t) = \frac{1}{2}\int_0^t (t-\tau)^2 \cos 2\tau\, d\tau \\
&= \frac{t}{4} - \frac{\sin 2t}{8},
\end{aligned}$$

where we have used Mathematica in the last line.

(c) (MP) Invert the Laplace transform directly using Mathematica.

3. (BH) Solve the following problem using Laplace transforms:

$$\ddot{y} + 16y = f(t), \qquad y(0) = a, \quad \dot{y}(0) = b.$$

Solution. Transforming the equation, we have

$$s^2\hat{y} - sa - b + 16\hat{y} = \hat{f}$$
$$\hat{y} = \frac{\hat{f} + as + b}{s^2 + 16} = \frac{\hat{f}}{4}\frac{4}{s^2+4^2} + a\frac{s}{s^2+4^2} + \frac{b}{4}\frac{4}{s^2+4^2},$$

where we have used the initial conditions. Since

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = \sin 4t, \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} = \cos 4t,$$

using the convolution theorem we have

$$y(t) = a\cos 4t + \frac{b}{4}\sin 4t + \frac{1}{4}\int_0^t f(t-\tau)\sin 4\tau\, d\tau.$$

4. (BH) Find the solution of the following Euler equations:

(a)

$$x^2y'' + 5xy' + 5y = 0.$$

Solution. This is an Euler equation, so we substitute $y = x^\lambda$ to obtain

$$\lambda(\lambda-1) + 5\lambda + 5 = \lambda^2 + 4\lambda + 5 = 0$$
$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i,$$
$$y = x^{-2\pm i} = x^{-2}e^{\pm i\log|x|}.$$

Therefore, we have that the general solution is given by

$$y(x) = x^{-2}[c_1\sin(\log|x|) + c_2\cos(\log|x|)].$$

(b)

$$x^2y'' + 5xy' + 4y = 0, \qquad y(1) = 1, \quad y'(1) = 1.$$

Solution. This is an Euler equation, so we substitute $y = x^\lambda$ to obtain

$$\lambda(\lambda-1) + 5\lambda + 4 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$$

Since we have a double root of $\lambda = -2$, by notes in class we have that the general solution is given by

$$y(x) = x^{-2}(c_1 + c_2\log|x|).$$

Solving the initial conditions, we have

$$y(1) = c_1 = 1, \qquad y'(1) = -2c_1 + c_2 = 1 \qquad \Longrightarrow \qquad c_2 = 3$$
$$y(x) = x^{-2}(1 + 3\log|x|).$$

5. (BH) Consider the following forced Euler equation:

$$ax^2y'' + bxy' + cy = x^\alpha.$$

(a) Try a particular solution of the form $y_\mathrm{p} = Ax^\alpha$, as in the method of undetermined coefficients. Explain why this approach works.

Solution. Substituting $y_\mathrm{p} = Ax^\alpha$ into our equation, we obtain

$$\begin{aligned} ax^2[A\alpha(\alpha-1)x^{\alpha-2}] + bx(A\alpha x^{\alpha-1}) + c(Ax^\alpha) &= x^\alpha \\ aA\alpha(\alpha-1) + bA\alpha + cA &= 1 \\ A &= \frac{1}{a\alpha(\alpha-1)+b\alpha+c}. \end{aligned}$$

This technique works because (just as in the homogeneous case) the coefficients of each derivative term multiply the derivative in just such a way that each term has an x^α in it.

(b) Use your approach in (a) to write the general solution of

$$x^2y'' - xy' - 3y = 6x^2.$$

Solution. We begin by solving the homogeneous problem. Substituting $y = x^\lambda$, we obtain

$$\begin{aligned} \lambda(\lambda-1) - \lambda + -3 = \lambda^2 - 2\lambda + 3 = (\lambda-3)(\lambda+1) = 0 \\ y_\mathrm{h} = c_1x^{-1} + c_2x^3. \end{aligned}$$

Next we solve for the particular solution. Letting $y_\mathrm{p} = Ax^2$, we have

$$\begin{aligned} x^2(2)(1)A - x(2)Ax - 3Ax^2 &= 6x^2 \\ -3A &= 6 \\ y_\mathrm{p} &= -2x^2 \\ y &= c_1x^{-1} + c_2x^3 - 2x^2. \end{aligned}$$

6. Consider the heat conduction problem

$$\frac{\partial^2 u}{\partial x^2} = 25\frac{\partial u}{\partial t}, \qquad 0 < x < \pi, \quad t > 0; \tag{9.1a}$$

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \quad t > 0; \tag{9.1b}$$

$$u(x,0) = \sin 3x + 2\sin 7x, \quad 0 \le x \le \pi. \tag{9.1c}$$

(a) (BH) Find the solution to (9.1).

Solution. Letting $u(x,t) = X(x)T(t)$ in (9.1a), we have

$$\begin{aligned} X''T &= 25T'X \\ \frac{X''}{X} &= 25\frac{T'}{T} = -\lambda^2, \end{aligned}$$

since the left-hand side is a function of x and the right-hand side is a function of t. Then the x-equation becomes

$$\begin{aligned} X'' + \lambda^2 X = 0, \qquad X(0) = X(\pi) = 0 \\ X(x) &= c_1 \sin \lambda x + c_2 \cos \lambda x \\ X(0) &= c_2 = 0 \\ X(\pi) &= c_1 \sin \lambda \pi = 0 \qquad \Longrightarrow \qquad \lambda = n, \quad n > 0, \end{aligned}$$

where we have used (9.1b). The initial conditions have $n = 3$ and $n = 7$, so by the principle of superposition we see that our solution is given by

$$u(x,t) = T_3(t) \sin 3x + T_7(t) \sin 7x, \qquad T_3(0) = 1, \quad T_7(0) = 2.$$

Substituting our results for λ_n into the T-equation, we obtain

$$\begin{aligned} 25T_n' + n^2 T_n = 0 \\ T_n(t) &= T_n(0) \exp\left(-n^2 t/25\right). \end{aligned}$$

Then substituting in our initial conditions with $n = 3$ and $n = 7$, we have

$$u(x,t) = \sin 3x \exp\left(-9t/25\right) + 2 \sin 7x \exp\left(-49t/25\right).$$

(b) (MP) Graph your solution for $t = 0$, 1, and 2. What happens to the number of oscillations as t gets larger? (*Hint: Look at the time dependence of your solution.*)

Solution. As t gets larger, the number of oscillations gets closer to 3. That is because the coefficient of the $\sin 7x$ term gets smaller much faster than the coefficient of the $\sin 3x$ term, and hence it is only the $\sin 3x$ term that we see.

7. (BH) The wave equation is given by

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

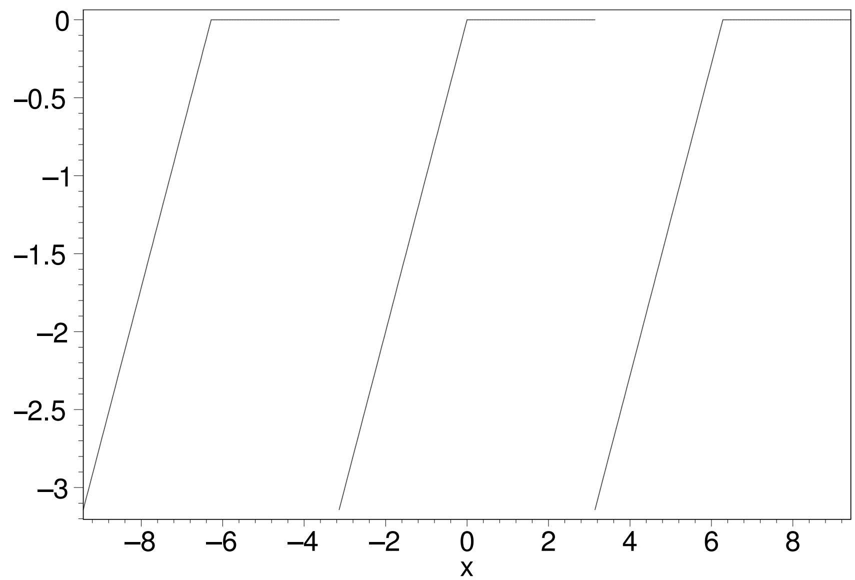
Assuming that $u(x,t) = T(t)X(x)$, find ordinary differential equations satisfied by T and X.

Solution. Letting $u(x,t) = X(x)T(t)$ in the above, we have

$$\begin{aligned} c^2 X'' T &= X T'' \\ c^2 \frac{X''}{X} &= \frac{T''}{T}. \end{aligned}$$

The left-hand side is a function of x; the right-hand side is a function of t. Therefore we must have that

$$\frac{T''}{T} = -\lambda^2 \qquad \Longrightarrow \qquad T'' + \lambda^2 T = 0$$



for some constant λ. Making this substitution into the above, we have

$$c^2 \frac{X''}{X} = -\lambda^2 \qquad \Longrightarrow \qquad c^2 X'' + \lambda^2 X = 0.$$

8. Consider the function

$$f(x) = \begin{cases} x, & -\pi < x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \qquad f(x + 2\pi) = f(x).$$

(a) (BH) Sketch the graph of the function for three periods.

Solution. See above.

(b) (BH) Find the Fourier series for the given function.

Solution. Here $L = \pi$, so we have

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(mx)\,dx = \frac{1}{\pi}\left[\frac{\cos(mx)}{m^2} + \frac{x\sin(mx)}{m}\right]_{-\pi}^{0}$$
$$= \frac{1}{\pi}\left[\frac{1-\cos(-m\pi)}{m^2}\right] = \frac{1-(-1)^m}{m^2\pi},$$

where we have used integration by parts. This term does not exist for $m = 0$, so we handle that case separately:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} = -\frac{\pi}{2},$$
$$b_m = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(mx)\,dx = \frac{1}{\pi}\left[\frac{\sin(mx)}{m^2} - \frac{x\cos(mx)}{m}\right]_{-\pi}^{0}$$
$$= \frac{1}{\pi}\left[\frac{-\pi\cos(-m\pi)}{m}\right] = -\frac{(-1)^m}{m}.$$

Since b_m is defined only for $m \geq 1$, we have no problems and our solution is given by

$$f(x) = -\frac{\pi}{4} + \sum_{m=1}^{\infty} \left[\frac{1-(-1)^m}{m^2\pi}\right] \cos mx - \frac{(-1)^m}{m} \sin mx,$$

where we have remembered to divide a_0 by 2. We note that the numerator of the bracketed term is zero if m is even and 2 if m is odd, so we may simplify the above to obtain

$$f(x) = -\frac{\pi}{4} + \sum_{m=1}^{\infty} \frac{2}{(2m-1)^2\pi} \cos(2m-1)x - \frac{(-1)^m}{m} \sin mx.$$

(c) (MP) Plot $s_m(x)$ (book notation for the first m terms in the Fourier series) for $m = 15$ and $x \in [-3\pi, 3\pi]$.

9. (MP) Find the first five terms in the Fourier series of

$$f(x) = xe^{-x}, \qquad x \in [-\pi, \pi].$$

10. (BH) Find the formal solution of the initial value problem

$$\ddot{y} + \omega^2 y = \sum_{n=1}^{\infty} b_n \sin nt, \qquad y(0) = 0, \qquad \dot{y}(0) = 0.$$

How is the solution altered if $\omega = m$, where m is a positive integer?

Solution. We use the method of undetermined coefficients. If the right-hand side were just $b_n \sin nt$, then by notes in class we know that we need try a solution of the form $y_{\mathrm{p},n} = B_n \sin nt$. Plugging this into the above, we obtain

$$\begin{aligned} -B_n n^2 \sin nt + \omega^2 B_n \sin nt &= b_n \sin nt \\ B_n &= \frac{b_n}{\omega^2 - n^2}, \quad n \neq \omega, \\ y_{\mathrm{p},n} &= \frac{b_n \sin nt}{\omega^2 - n^2}, \quad n \neq \omega. \end{aligned}$$

However, if $\omega = m$, then the right-hand side is a solution to the homogeneous operator. Thus we must try a soltion of the form $y_{\mathrm{p},m} = t(A_m \cos mt + B_m \sin mt)$. Substituting this into the above with $\omega = m$, we have

$$\begin{aligned} &\frac{d}{dt}\left[mt(-A_m \sin mt + B_m \cos mt) + A_m \cos mt + B_m \sin mt\right] \\ &\qquad + m^2 t(A_m \cos mt + B_m \sin mt) = b_m \sin mt \\ &-m^2 t(A_m \cos mt + B_m \sin mt) + 2m(-A_m \sin mt + B_m \cos mt) \\ &\qquad + m^2 t(A_m \cos mt + B_m \sin mt) = b_m \sin mt \end{aligned}$$

$$B_m = 0, \qquad A_m = -\frac{b_m}{2m}$$
$$y_{\mathrm{p},m} = -\frac{b_m t \cos mt}{2m}.$$

But then by the principle of superposition we know that the full particular solution is just the sum of each of the $y_{\mathrm{p},n}$, so we have

$$y_{\mathrm{p}}(t) = \sum_{n=1}^{\infty} \frac{b_n \sin nt}{\omega^2 - n^2}, \qquad \omega \neq m,$$
$$y_{\mathrm{p}}(t) = -\frac{b_m t \cos mt}{2m} + \sum_{n \neq m} \frac{b_n \sin nt}{\omega^2 - n^2}, \qquad \omega = m.$$

We note that in both cases we have $y_{\mathrm{p}}(0) = 0$. The homogeneous solutions are given by $\sin \omega t$, $\cos \omega t$. Thus we have

$$y(t) = a_1 \sin \omega t + a_2 \cos \omega t + y_{\mathrm{p}}(t)$$
$$y(0) = a_2 + y_{\mathrm{p}}(0) = a_2 = 0$$
$$\dot{y}(0) = \omega a_1 + \dot{y}_{\mathrm{p}}(0) = 0$$
$$0 = \omega a_1 + \sum_{n=1}^{\infty} \frac{n b_n}{\omega^2 - n^2}$$
$$a_1 = -\frac{1}{\omega} \sum_{n=1}^{\infty} \frac{n b_n}{\omega^2 - n^2}$$
$$y(t) = \sum_{n=1}^{\infty} \frac{b_n}{\omega^2 - n^2} \left(\sin nt - \frac{n}{\omega} \sin \omega t \right), \qquad \omega \neq m.$$

For the case where $\omega = m$, we note that

$$\dot{y}_{\mathrm{p},m}(0) = -\frac{b_m}{2m}.$$

Thus to cancel that out we have

$$y(t) = -\frac{b_m}{2m} \left(t \cos mt - \frac{\sin mt}{m} \right) + \sum_{n \neq m} \frac{b_n}{m^2 - n^2} \left(\sin nt - \frac{n}{m} \sin mt \right), \qquad \omega = m.$$

In[ ]:=
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Quit[]
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HW1 (Checked)

HW2 (Checked)

HW3 (Checked)

HW4 (Checked)

HW5 (Checked)

HW6 (Checked)

HW7 (Checked)

HW8 (Revised 11/19)

HW9 (Checked)

Number 2c.

In[1]:=
```
lap7c = 1 / s^2 / (s^2 + 4)
InverseLaplaceTransform[lap7c, s, t]
```

Out[1]= $\frac{1}{s^2 \left(4+s^2\right)}$

Out[2]= $\frac{1}{4} (t - \text{Cos}[t]\ \text{Sin}[t])$

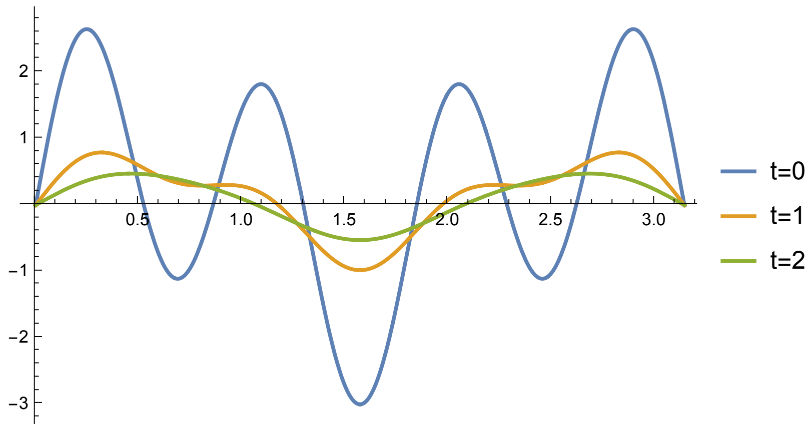
Number 6b.

In[3]:=
```
sol9 = Sin[3 * x] * Exp[-9 * t / 25] + 2 * Sin[7 * x] * Exp[-49 * t / 25]
tab9 = Table[sol9, {t, 0, 2, 1}]
Plot[tab9, {x, 0, Pi}, PlotLegends → {"t=0", "t=1", "t=2"}]
```

Out[3]= $e^{-9t/25} \text{Sin}[3x] + 2 e^{-49t/25} \text{Sin}[7x]$

Out[4]= $\left\{\text{Sin}[3x] + 2\,\text{Sin}[7x],\ \frac{\text{Sin}[3x]}{e^{9/25}} + \frac{2\,\text{Sin}[7x]}{e^{49/25}},\ \frac{\text{Sin}[3x]}{e^{18/25}} + \frac{2\,\text{Sin}[7x]}{e^{98/25}}\right\}$

Out[5]=



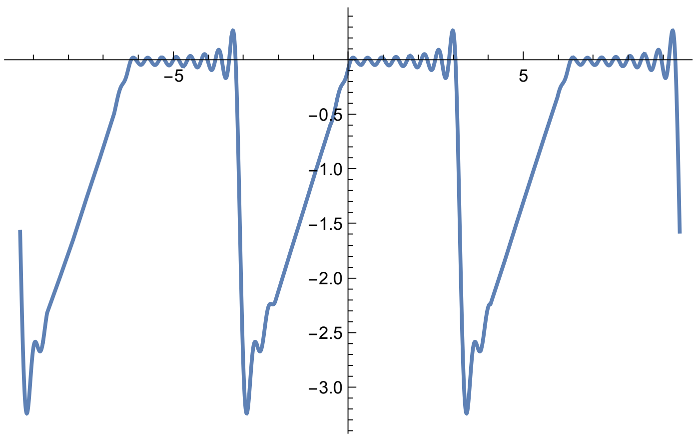
Number 8c.

In[6]:=
```
sm = -Pi / 4 +
  Sum[2 * Cos[(2 * m - 1) * x] / Pi / (2 * m - 1) ^ 2 + (-1) ^ (m + 1) * Sin[m * x] / m, {m, 1, 15}]
Plot[sm, {x, -3 * Pi, 3 * Pi}]
```

Out[6]=

$$-\frac{\pi}{4}+\frac{2\,\mathrm{Cos}[x]}{\pi}+\frac{2\,\mathrm{Cos}[3\,x]}{9\,\pi}+\frac{2\,\mathrm{Cos}[5\,x]}{25\,\pi}+\frac{2\,\mathrm{Cos}[7\,x]}{49\,\pi}+\frac{2\,\mathrm{Cos}[9\,x]}{81\,\pi}+\frac{2\,\mathrm{Cos}[11\,x]}{121\,\pi}+\frac{2\,\mathrm{Cos}[13\,x]}{169\,\pi}+\frac{2\,\mathrm{Cos}[15\,x]}{225\,\pi}+\frac{2\,\mathrm{Cos}[17\,x]}{289\,\pi}+\frac{2\,\mathrm{Cos}[19\,x]}{361\,\pi}+\frac{2\,\mathrm{Cos}[21\,x]}{441\,\pi}+\frac{2\,\mathrm{Cos}[23\,x]}{529\,\pi}+\frac{2\,\mathrm{Cos}[25\,x]}{625\,\pi}+\frac{2\,\mathrm{Cos}[27\,x]}{729\,\pi}+\frac{2\,\mathrm{Cos}[29\,x]}{841\,\pi}+\mathrm{Sin}[x]-\frac{1}{2}\mathrm{Sin}[2\,x]+\frac{1}{3}\mathrm{Sin}[3\,x]-\frac{1}{4}\mathrm{Sin}[4\,x]+\frac{1}{5}\mathrm{Sin}[5\,x]-\frac{1}{6}\mathrm{Sin}[6\,x]+\frac{1}{7}\mathrm{Sin}[7\,x]-\frac{1}{8}\mathrm{Sin}[8\,x]+\frac{1}{9}\mathrm{Sin}[9\,x]-\frac{1}{10}\mathrm{Sin}[10\,x]+\frac{1}{11}\mathrm{Sin}[11\,x]-\frac{1}{12}\mathrm{Sin}[12\,x]+\frac{1}{13}\mathrm{Sin}[13\,x]-\frac{1}{14}\mathrm{Sin}[14\,x]+\frac{1}{15}\mathrm{Sin}[15\,x]$$

Out[7]=



Number 9.

In[8]:=
```
f2 = x * Exp[-x]
FourierTrigSeries[f2, x, 5]
```

Out[8]= $e^{-x}\, x$

Out[9]=
$$-\frac{e^{\pi}\,(-1+\pi)+e^{-\pi}\,(1+\pi)}{2\,\pi}+\text{Cos}[x]\,\text{Cosh}[\pi]+\frac{\text{Sin}[5\,x]\,(65\,\pi\,\text{Cosh}[\pi]-5\,\text{Sinh}[\pi])}{169\,\pi}+\frac{8\,\text{Sin}[4\,x]\,(-17\,\pi\,\text{Cosh}[\pi]+2\,\text{Sinh}[\pi])}{289\,\pi}-\frac{2\,\text{Cos}[2\,x]\,(5\,\pi\,\text{Cosh}[\pi]+3\,\text{Sinh}[\pi])}{25\,\pi}-\frac{\text{Cos}[4\,x]\,(68\,\pi\,\text{Cosh}[\pi]+60\,\text{Sinh}[\pi])}{578\,\pi}+\frac{1}{2}\,\text{Sin}[x]\left(2\,\text{Cosh}[\pi]-\frac{2\,\text{Sinh}[\pi]}{\pi}\right)+\frac{1}{2}\,\text{Sin}[3\,x]\left(\frac{6\,\text{Cosh}[\pi]}{5}-\frac{6\,\text{Sinh}[\pi]}{25\,\pi}\right)+\frac{1}{2}\,\text{Cos}[5\,x]\left(\frac{2\,\text{Cosh}[\pi]}{13}+\frac{24\,\text{Sinh}[\pi]}{169\,\pi}\right)+\frac{1}{2}\,\text{Cos}[3\,x]\left(\frac{2\,\text{Cosh}[\pi]}{5}+\frac{8\,\text{Sinh}[\pi]}{25\,\pi}\right)+\frac{4}{25}\,\text{Sin}[2\,x]\left(-5\,\text{Cosh}[\pi]+\frac{2\,\text{Sinh}[\pi]}{\pi}\right)$$

SSM (Checked)