Homework Set 7 Solutions (11/19 Version)

1. (BH) Consider the matrix

$$B = \begin{pmatrix} 4 & 3 \\ -6 & -2 \end{pmatrix}.$$

(a) Find the eigenvalues of B.

Solution. Calculating the characteristic polynomial, we have

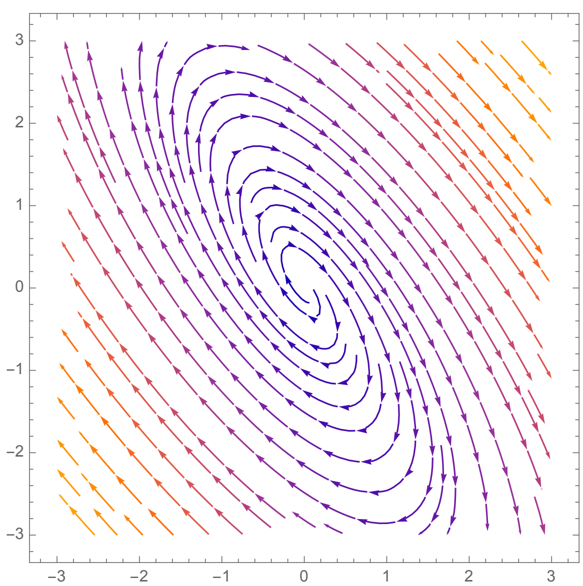
$$\begin{vmatrix} 4-\lambda & 3 \\ -6 & -2-\lambda \end{vmatrix} = -(4-\lambda)(2+\lambda) + 18 = \lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i.$$

(b) Classify the fixed point at the origin.

Solution. Because the real part of λ is positive, we have an unstable spiral.

(c) Sketch the phase plane for the system $\dot{\mathbf{x}} = B\mathbf{x}$.



2. (BH) Find the solution of

$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

You should express your answers in terms of real functions.

Solution. Calculating the characteristic polynomial, we have

$$\begin{vmatrix} -3-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = -(3+\lambda)(1-\lambda)+8 = \lambda^2+2\lambda+5=0$$
$$\lambda = \frac{-2\pm\sqrt{4-20}}{2} = -1\pm 2i.$$

Now we must calculate the eigenvectors corresponding to the eigenvalues. Solving for the first eigenvector, we obtain

$$(A-(-1+2i)I)\mathbf{z}_+ = \begin{pmatrix} -2-2i & 2 \\ -4 & 2-2i \end{pmatrix}\mathbf{z}_1 = \mathbf{0}.$$

The second row is $1-i$ times the first row, so the equations are redundant and from the first row we have

$$2y = 2(1+i)x \qquad \Longrightarrow \qquad x=1, y=1+i.$$

Hence we take the real and imaginary parts of the following product:

$$\begin{aligned} e^{(-1+2i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} &= e^{-t}\left[(\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right] \\ &= e^{-t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}. \end{aligned}$$

Therefore, our general solution is

$$\mathbf{x}(t) = c_1e^{-t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + c_2e^{-t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}. \tag{A}$$

Substituting $t=0$ in (A) to find the initial conditions, we have

$$\begin{aligned} c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ c_1 &= 4 \\ c_1+c_2 &= 3 \qquad \Longrightarrow \qquad c_2 = -1 \\ \mathbf{x}(t) &= 4e^{-t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} - e^{-t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} 4\cos 2t - \sin 2t \\ 3\cos 2t - 5\sin 2t \end{pmatrix}. \end{aligned}$$

3. Consider the matrix

$$\dot{\mathbf{x}} = \begin{pmatrix} 6 & -8 \\ 2 & -2 \end{pmatrix}\mathbf{x}. \tag{7.1}$$

   (a) (BH) Write the general solution of (7.1).

Solution. Calculating the characteristic polynomial, we have

$$\begin{vmatrix} 6-\lambda & -8 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda-6)(\lambda+2)+16 = \lambda^2-4\lambda+4 = (\lambda-2)^2 = 0.$$

Thus we have a repeated eigenvalue, so our solution is of the form

$$\mathbf{x}(t) = c_1 e^{2t}\mathbf{z}_1 + c_2 e^{2t}(t\mathbf{z}_1 + \vec{\eta}).$$

Now we must calculate the eigenvectors corresponding to the eigenvalues. Solving for the first eigenvector, we obtain

$$(A-2I)\mathbf{z}_1 = \begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix}\mathbf{z}_1 = \mathbf{0}.$$

We note that the second row is a multiple of the first, so the equations are redundant. Thus we must solve $2x-4y=0$, so a typical eigenvector is $\mathbf{z}_1 = (2,1)$. Solving for the generalized eigenvector $\vec{\eta}$, we obtain

$$\begin{aligned} (A-2I)\vec{\eta} &= \begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix}\vec{\eta} = \mathbf{z}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ 4x-8y &= 2 \\ 2x-4y &= 1 \end{aligned}$$

We note that the second equation is a multiple of the first, so they are redundant. Thus we have that $2x-4y=1$, so a typical eigenvector is $\mathbf{z}_2 = (1/2, 0)$. Thus our solution is of the form

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t}\left[t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right].$$

(b) (MP) Sketch the phase plane for (7.1).

4. (BH) Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{7.2}$$

(a) Write the solution of (7.2) for $\epsilon \neq 0$.

Solution. This is a diagonal matrix, so the eigenvalues are the diagonal entries, and $\lambda = 0$ is a repeated eigenvalue. Thus we have a repeated eigenvalue, so our solution should be of the form

$$\mathbf{x}(t) = c_1\mathbf{z}_1 + c_2(t\mathbf{z}_1 + \vec{\eta}).$$

Now we must calculate the eigenvectors corresponding to the eigenvalues. Solving for the first eigenvector, we obtain

$$A\mathbf{z}_1 = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}\mathbf{z}_1 = \mathbf{0}.$$

Obviously the equations are redundant. Thus we must solve $-\epsilon y = 0$, so a typical eigenvector is $\mathbf{z}_1 = (1, 0)$. Solving for the generalized eigenvector $\vec{\eta}$, we obtain

$$\begin{aligned} A\vec{\eta} &= \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix} \vec{\eta} = \mathbf{z}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \epsilon y &= 1 \\ 0 &= 0 \end{aligned}$$

Therefore, a typical eigenvector is $\vec{\eta} = (0, \epsilon^{-1})$. Thus our solution is of the form

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \left[t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \epsilon^{-1} \end{pmatrix} \right].$$

Solving the initial conditions, we have

$$\begin{aligned} \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} c_1 \\ \epsilon^{-1} c_2 \end{pmatrix} \\ c_1 = 1, &\quad c_2 = \epsilon, \\ \mathbf{x}(t) &= \begin{pmatrix} 1 \\ 1 + \epsilon t \end{pmatrix}. \end{aligned} \tag{B}$$

(b) Write the solution of (7.2) for $\epsilon = 0$.

Solution. If $\epsilon = 0$, we have $\dot{\mathbf{x}} = \mathbf{0}$, which means that $\mathbf{x}$ never changes, so $\mathbf{x}(t) = (1, 1)$.

(c) Show that if you take the limit of your answer to (a) for $\epsilon \to 0$, you get (b).

Solution. Letting $\epsilon \to 0$ in (B), we have $\mathbf{x}(t) = (1, 1)$, as required.

5. (BH) Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ e^{2t} \end{pmatrix}. \tag{7.3}$$

Using the method of undetermined coefficients, show that a particular solution of this problem is given by

$$\mathbf{x}_\mathrm{p} = \frac{1}{3} \begin{pmatrix} 3 - e^{2t} \\ -12 - e^{2t} \end{pmatrix}.$$

Solution. We write our particular solution as

$$\mathbf{x}_\mathrm{p} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + e^{2t} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Plugging this formula into (7.3), we obtain

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 2e^{2t} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + e^{2t} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right] + \begin{pmatrix} 3 \\ e^{2t} \end{pmatrix} \\ 2e^{2t} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} a_1 + a_2 + 3 \\ 4a_1 + a_2 \end{pmatrix} + e^{2t} \begin{pmatrix} b_1 + b_2 \\ 4b_1 + b_2 + 1 \end{pmatrix}, \end{aligned}$$

where we have grouped by function of time. Balancing the constant equations (where there is nothing on the right-hand side), we have

$$\begin{aligned} 0 &= a_1 + a_2 + 3 \\ 0 &= 4a_1 + a_2 \end{aligned} \qquad \Longrightarrow \qquad a_1 = 1, \quad a_2 = -4.$$

Balancing the e^{2t} equations, we have

$$\begin{aligned} 2b_1 &= b_1 + b_2 \\ 2b_2 &= 4b_1 + b_2 + 1 \end{aligned} \qquad \Longrightarrow \qquad b_1 = b_2 = -\frac{1}{3}.$$

Thus our solution is given by

$$\mathbf{x}_\mathrm{p} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} + e^{2t} \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 - e^{2t} \\ -12 - e^{2t} \end{pmatrix},$$

as required.

6. Consider the system

$$\dot{x} = xy, \tag{7.4a}$$
$$\dot{y} = y - x^2 + 1. \tag{7.4b}$$

(a) (BH) Find and characterize all the fixed points.

Solution. Setting the right-hand side of (7.4a) equal to zero, we have that either $x = 0$ or $y = 0$. Substituting $y = 0$ into the right-hand side of (7.4b) and setting equal to zero, we obtain $1 - x^2 = 0$, so we have $(\pm 1, 0)$ as fixed points. Similarly, substituting $x = 0$ into the right-hand side of (7.4b) and setting equal to zero, we obtain $y + 1 = 0$, so the final fixed point is $(0, -1)$.

Calculating the Jacobian in general, we have

$$J(x, y) = \begin{pmatrix} y & x \\ -2x & 1 \end{pmatrix}.$$

Then evaluating the Jacobian at each of the fixed points, we have

$$J(0, -1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad \lambda = -1,\, 1,$$

so $(0, -1)$ is a saddle. Continuing, we have

$$J(1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}$$
$$\begin{vmatrix} -\lambda & 1 \\ -2 & 1 - \lambda \end{vmatrix} = \lambda(\lambda - 1) + 2 = \lambda^2 - \lambda + 2 \qquad \Longrightarrow \qquad \lambda = \frac{1 \pm \sqrt{1 - 8}}{2}.$$

Therefore, $(1,0)$ is an unstable spiral. Examining the last fixed point, we have

$$J(-1,0) = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = \lambda(\lambda-1)+2 = \lambda^2 - \lambda + 2,$$

so $(-1,0)$ has the same eigenvalues as $(1,0)$ and hence is also an unstable spiral.

(b) (MP) Sketch the phase plane.

7. (BH) For the two systems below, characterize the fixed point at the origin, and discuss what might happen to the fixed point if nonlinear terms are added to the equations.

$$\text{(a)}: \quad \begin{aligned} \dot{x} &= -2x - y \\ \dot{y} &= 5x + 2y \end{aligned}$$

Solution. Rewriting in matrix form, we have $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}.$$

The eigenvalues are given by

$$\begin{vmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = (\lambda-2)(\lambda+2)+5 = \lambda^2+1 = 0.$$

Therefore, $\lambda = \pm i$ and the origin is a center. With nonlinear terms, the origin can remain a center or become a spiral.

$$\text{(b)}: \quad \begin{aligned} \dot{x} &= -4x - y \\ \dot{y} &= x - 2y \end{aligned}$$

Solution. Rewriting in matrix form, we have $\dot{\mathbf{x}} = B\mathbf{x}$, where

$$B = \begin{pmatrix} -4 & -1 \\ 1 & -2 \end{pmatrix}.$$

The eigenvalues are given by

$$\begin{vmatrix} -4-\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+4)(\lambda+2)+1 = \lambda^2+6\lambda+9 = (\lambda+3)^2 = 0.$$

Therefore, we have an improper stable node. With nonlinear terms, the origin can become a regular stable node or a stable spiral.

8. Consider the system

$$\dot{x} = x(x^2 + \mu) - 3y, \tag{7.5a}$$
$$\dot{y} = x^2 y + 3x. \tag{7.5b}$$

(a) (BH) Show that (7.5) exhibits a Hopf bifurcation as μ passes through zero.

Solution. Changing into polar coordinates, we calculate

$$\frac{d(r^2)}{dt} = \frac{d(x^2 + y^2)}{dt} = 2x\dot{x} + 2y\dot{y}.$$

Substituting (7.5) into the above, we have

$$\begin{aligned} \frac{d(r^2)}{dt} &= 2x[x(x^2 + \mu) - 3y] + 2y[x^2 y + 3x] = 2x^2(x^2 + y^2 + \mu) \\ 2r\dot{r} &= 2(r^2 \cos^2\theta)(r^2 + \mu) \\ \dot{r} &= r(\cos^2\theta)(r^2 + \mu). \end{aligned}$$

Therefore, we see that $r = 0$ (the origin) is always a fixed point. If $\mu > 0$, then $\dot{r} > 0$ for all r and the origin is unstable. However, if $\mu < 0$, then there is a limit cycle with $r = \sqrt{-\mu}$, and the origin becomes stable.

(b) (MP) Sketch one phase plane for the system for μ positive and one for μ negative. Make sure that the axes are large enough to illustrate the limit cycle.

9. (BH) Calculate the Laplace transform of the following functions. Use the *definition*, not the table. For what range of s will the transforms exist?

(a) $\cosh \omega t$

Solution.

$$\begin{aligned} \mathcal{L}(\cosh \omega t) &= \int_0^\infty e^{-st} \cosh \omega t \, dt = \int_0^\infty e^{-st} \left(\frac{e^{\omega t} + e^{-\omega t}}{2} \right) dt \qquad \text{(C.1)} \\ &= \frac{1}{2} \int_0^\infty e^{-(s-\omega)t} + e^{-(s+\omega)t} \, dt = \frac{1}{2} \left[\frac{e^{-(s-\omega)t}}{-(s-\omega)} + \frac{e^{-(s+\omega)t}}{-(s+\omega)} \right]_0^\infty \\ &= \frac{1}{2} \left(\frac{1}{s-\omega} + \frac{1}{s+\omega} \right) = \frac{s}{s^2 - \omega^2}. \qquad \text{(C.2)} \end{aligned}$$

For this to exist, $s - \omega > 0$ and $s + \omega > 0$, so $s > |\omega|$.

(b) $\sinh \omega t$. In this case,

$$\mathcal{L}\{\sinh \omega t\} = \int_0^\infty e^{-st} \sinh \omega t \, dt. \tag{7.6}$$

Solution.

$$\mathcal{L}(\sinh \omega t) = \int_0^\infty e^{-st} \sinh \omega t \, dt = \int_0^\infty e^{-st} \left(\frac{e^{\omega t} - e^{-\omega t}}{2} \right) dt.$$

But this is just (C.1) with a minus sign between the terms instead of a plus, so we make the same adjustment to (C.2):

$$\mathcal{L}(\sinh \omega t) = \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) = \frac{\omega}{s^2-\omega^2}.$$

For this to exist, $s-\omega > 0$ and $s+\omega > 0$, so $s > |\omega|$.

(c) $t \sinh \omega t$.

Solution.

$$\begin{aligned}\mathcal{L}(t \sinh \omega t) &= \int_0^\infty e^{-st} t \sinh \omega t \, dt = -\int_0^\infty \frac{\partial(e^{-st})}{\partial s} \sinh \omega t \, dt = -\frac{\partial(\mathcal{L}(\sinh \omega t))}{\partial s} \\ &= -\frac{\partial}{\partial s}\left(\frac{\omega}{s^2-\omega^2}\right) = \frac{2s\omega}{(s^2-\omega^2)^2},\end{aligned}$$

where we have used (7.6). For this to exist, $s-\omega > 0$ and $s+\omega > 0$, so $s > |\omega|$.

10. (MP) Calculate either the Laplace transform or the inverse Laplace transform of the following functions:

$$\begin{aligned}
&(a): && \hat{y}(s) = \frac{1}{s^4-a^4} \\
&(b): && y(t) = t\sin(2\sqrt{t}) \\
&(c): && y(t) = \frac{e^{-1/t}}{\sqrt{t}} \\
&(d): && \hat{y}(s) = \frac{e^{-1/s}}{\sqrt{s}}
\end{aligned}$$

Number 3b.

```
b3 = {{6, -8}, {2, -2}}
field3 = b3.{x, y}
StreamPlot[field3, {x, -3, 3}, {y, -3, 3}]
```

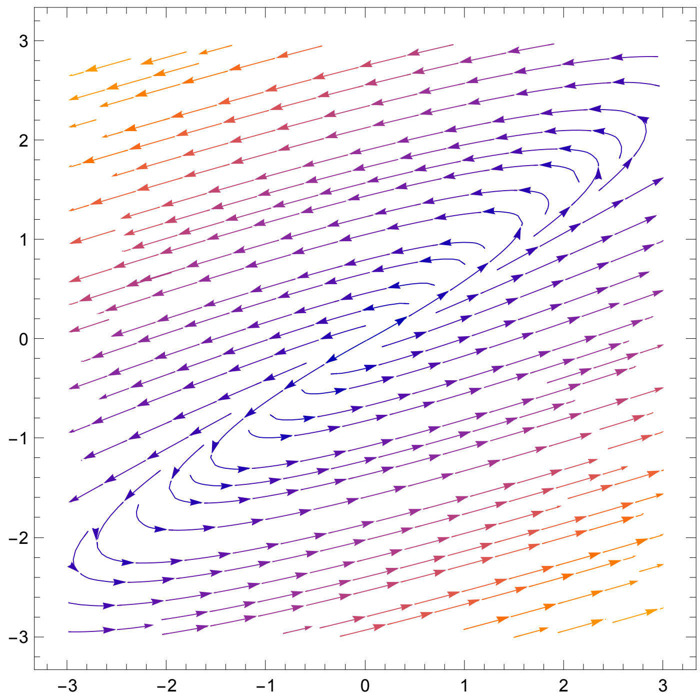
Out[∘]=

{{6, -8}, {2, -2}}

Out[∘]=

{6 x - 8 y, 2 x - 2 y}

Out[∘]=



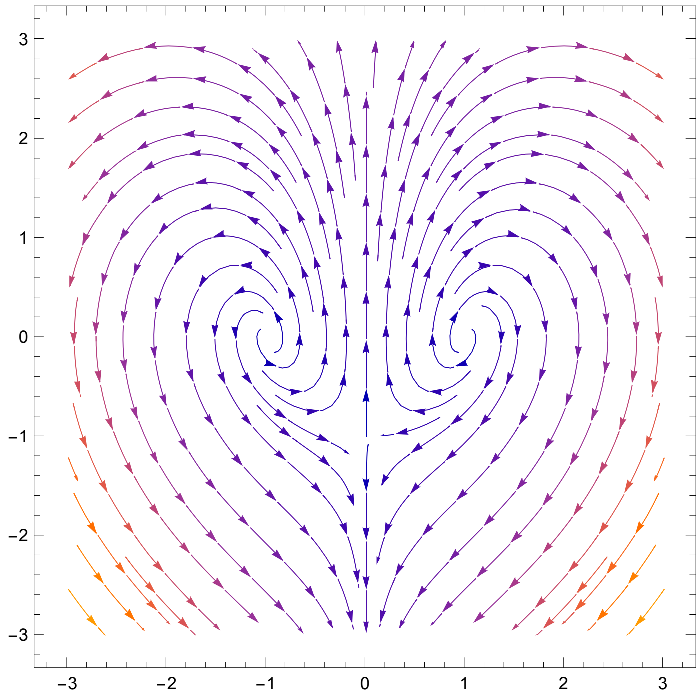
Number 6b.

In[ ]:=
```
field6 = {x * y, y - (x) ^ 2 + 1}
StreamPlot[field6, {x, -3, 3}, {y, -3, 3}]
```

Out[ ]=

$\{x\, y, 1 - x^2 + y\}$

Out[ ]=



Number 8b.

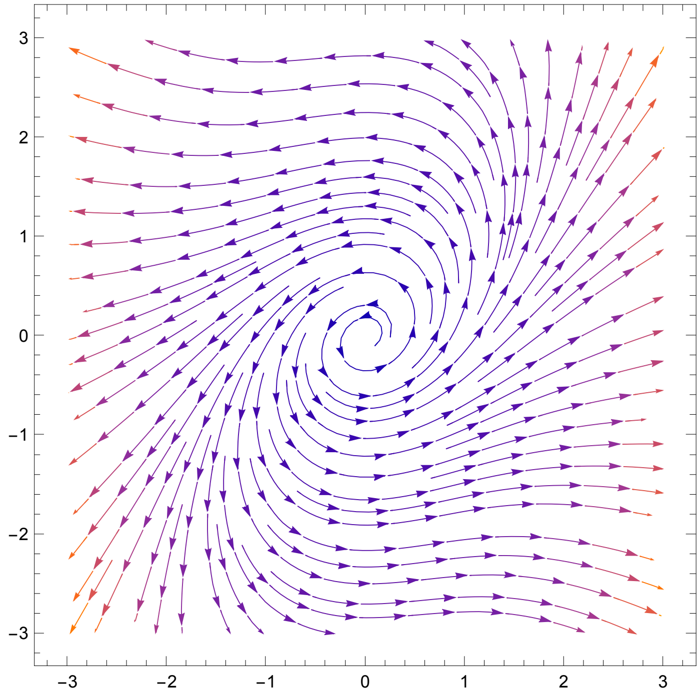
In[ ]:=
```
vecsys8 = {x * (x ^ 2 + mu) - 3 * y, x ^ 2 * y + 3 * x}
```

Out[ ]=

$\{x\,(mu + x^2) - 3\,y, 3\,x + x^2\,y\}$

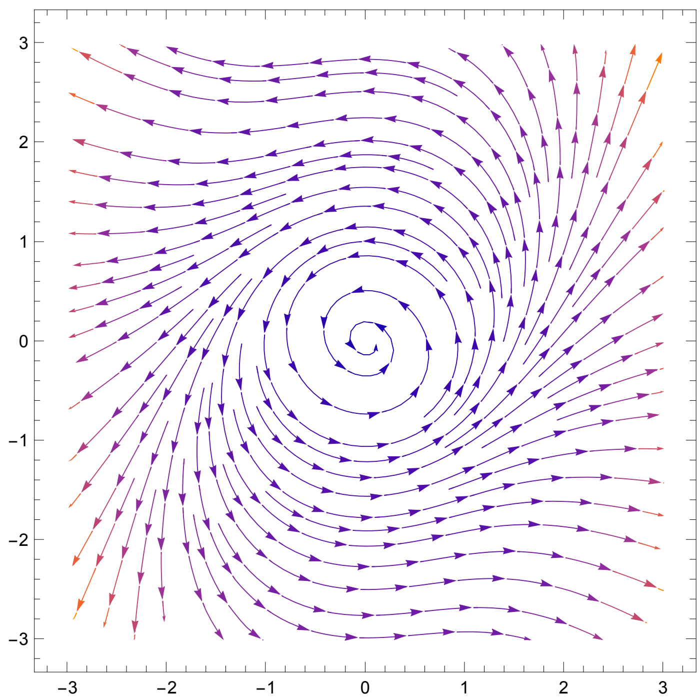
In[ ]:= StreamPlot[vecsys8 /. (mu → 1), {x, -3, 3}, {y, -3, 3}]

Out[ ]=



In[ ]:= StreamPlot[vecsys8 /. (mu → -1), {x, -3, 3}, {y, -3, 3}]

Out[ ]=



Number 10a.

In[∘]:=
```
ques10a = 1 / (s^4 - a^4)
InverseLaplaceTransform[ques10a, s, t]
```

Out[∘]=

$$\frac{1}{-a^4+s^4}$$

Out[∘]=

$$-\frac{e^{-a t}-e^{a t}+2 \operatorname{Sin}[a t]}{4 a^3}$$

Number 10b.

In[∘]:=
```
ques10b = t * Sin[2 * Sqrt[t]]
LaplaceTransform[ques10b, t, s]
```

Out[∘]=

$$t \operatorname{Sin}\left[2 \sqrt{t}\right]$$

Out[∘]=

$$\frac{e^{-1/s} \sqrt{\pi}\ (-2+3 s)}{2 s^{7/2}}$$

Number 10c.

In[∘]:=
```
ques10c = Exp[-1 / t] / Sqrt[t]
LaplaceTransform[ques10c, t, s]
```

Out[∘]=

$$\frac{e^{-1/t}}{\sqrt{t}}$$

Out[∘]=

$$\frac{e^{-2 \sqrt{s}} \sqrt{\pi}}{\sqrt{s}}$$

Number 10d.

In[∘]:=
```
ques10d = Exp[-1 / s] / Sqrt[s]
InverseLaplaceTransform[ques10d, s, t]
```

Out[∘]=

$$\frac{e^{-1/s}}{\sqrt{s}}$$

Out[∘]=

$$\frac{\operatorname{Cosh}\left[2 \sqrt{-t}\right]}{\sqrt{\pi} \sqrt{t}}$$