Homework Set 5 Solutions

1. A mass is attached to a vertical spring which has internal damping. The relevant equation is given by
$$\ddot{x} + \frac{1}{32}\dot{x} + 96x = 0, \tag{5.1}$$
where x is displacement measured (in cm) from the rest position, and time is measured in seconds. The spring is stretched 1/7 cm and then released.

   (a) (BH) What are the initial conditions that correspond to this situation?

Solution. The mass is stretched 1/7 cm, so $x(0) = 1/7$. The spring is simply released without additional velocity, so $\dot{x}(0) = 0$.

   (b) (MP) Solve the corresponding system.

   (c) (MP) Equipment in the lab can measure the displacement down to a level of 0.01 cm. Estimate the time t_* after which the displacement of the spring *always* remain below the threshold level.

2. (BH) The displacement $x(t)$ of a spring is governed by the following equation:
$$9\ddot{x} + 12\dot{x} + 4x = 0, \qquad x(0) = x_0, \quad \dot{x}(0) = v_0.$$

   (a) Construct the solution to this problem.

Solution. Substituting $x = e^{\lambda t}$, we obtain
$$9\lambda^2 + 12\lambda + 4 = (3\lambda + 2)^2 = 0 \qquad \Longrightarrow \qquad \lambda = -2/3,$$
so we have a double root. Therefore, our solutions are of the form $x = e^{-2t/3}(c_1 + c_2 t)$. Solving the first initial condition, we immediately have that $c_1 = x_0$. Solving the second initial condition, we have
$$\begin{aligned} e^{-2t/3}(c_2) - \frac{2e^{-t}}{3}(x_0 + c_2 t)\bigg|_{t=0} &= v_0 \\ c_2 - \frac{2x_0}{3} &= v_0 \\ x(t) &= e^{-2t/3}\left[x_0 + \left(v_0 + \frac{2x_0}{3}\right)t\right]. \end{aligned}$$

   (b) Show that $x(t_*) = 0$ if and only if $v_0/x_0 < -2/3$. In this case, how many times does the solution cross the t-axis?

Solution. The solution is zero whenever

$$x_0 + \left(v_0 + \frac{2x_0}{3}\right) t = 0$$

$$t = -\frac{x_0}{v_0 + 2x_0/3} = -\frac{1}{v_0/x_0 + 2/3} > 0. \qquad \text{(A)}$$

For the last inequality in (A) to be true, we must have

$$\frac{v_0}{x_0} + \frac{2}{3} < 0,$$

from which the result follows. Note that there is only one solution for t in (A), and hence if the solution crosses the t-axis once, the solution crosses it *only* once.

3. In steady state, the temperature $T(x)$ in a domain obeys the following equation:

$$T'' - VT' = 0, \qquad T(0) = 0, \quad T(1) = 1, \qquad (5.2)$$

where V is the velocity of heat flow.

(a) (BH) Solve (5.2) for $V \neq 0$.

Solution. Substituting $T = e^{\lambda x}$, we obtain

$$\lambda^2 + V\lambda = \lambda(\lambda - V) = 0$$

$$T(x) = c_1 + c_2 e^{Vx}$$

$$T(0) = c_1 + c_2 = 0 \implies T(x) = c_1(1 - e^{Vx})$$

$$T(1) = c_1(1 - e^V) = 1 \implies T(x) = \frac{1 - e^{Vx}}{1 - e^V}. \qquad \text{(B)}$$

(b) (BH) By taking the limit of your answer to (a), solve (5.2) for $V = 0$.

Solution. Using l'Hôpital's Rule for the limit of (B) as $V \to 0$, we have

$$T(x) = \lim_{V\to 0} \frac{1 - e^{Vx}}{1 - e^V} = \lim_{V\to 0} \frac{-xe^{Vx}}{-e^V} = x.$$

(c) (MP) Graph your solution for $V = -2$, 0, and 2. Interpret your solution in terms of the velocity.

4. (BH) Under a certain model, the amount of income $Y(t)$ in an economy obeys the following equation:

$$\ddot{Y} + 4\dot{Y} + (3 + \alpha)Y = -1; \qquad \alpha > 0, \quad \alpha \neq 1. \qquad (5.3)$$

(a) Find the general solution of (5.3), as well as the steady state of Y.

Solution. To obtain the homogeneous solution, we try $Y_{\text{h}} = e^{\lambda t}$, which yields

$$\begin{aligned}
&\lambda^2 + 4\lambda + (3+\alpha) = 0, \\
&\qquad\qquad \lambda_{\pm} = -\frac{4 \pm \sqrt{16 - 4(3+\alpha)}}{2} = -\frac{4 \pm 2\sqrt{1-\alpha}}{2} = -2 \pm \sqrt{1-\alpha} \\
&\qquad\qquad Y_{\text{h}} = Ae^{\lambda_+ t} + Be^{\lambda_- t}.
\end{aligned}$$

Using the method of undetermined coefficients, we try to find a particular solution of the form

$$Y_{\text{p}} = C.$$

Substituting in this form, we obtain

$$\begin{aligned}
&(3+\alpha)CY = -1 \\
&\qquad Y_{\text{p}} = -\frac{1}{3+\alpha} \\
&\qquad Y(t) = -\frac{1}{3+\alpha} + Ae^{\lambda_+ t} + Be^{\lambda_- t}.
\end{aligned}$$

Since $\alpha > 0$, the square root in λ is never larger than 1, so both roots always have negative real part. Hence $e^{\lambda_{\pm} t} \to 0$ as $t \to \infty$, and hence the steady state is given by the particular solution:

$$Y_{\text{s}}(t) = -\frac{1}{3+\alpha}. \tag{C}$$

(b) For what values of α will the solution oscillate?

Solution. The solutions oscillate when λ is complex, which occurs when the discriminant is negative, or when $\alpha > 1$.

(c) Economists would like to maximize the steady state of Y while keeping the income from oscillating. Are those two goals compatible for this model?

Solution. Since the steady state is negative, we see from (C) that maximizing the steady state involves driving α as large as possible. But for any $\alpha > 1$, income oscillates, so the two goals are in conflict.

5. We reconsider the damped spring of #1, but impose a forcing $F(t)$:

$$\ddot{x} + \frac{1}{32}\dot{x} + 96x = F(t),$$

where

$$F(t) = \begin{cases} 4\sin t, & 0 \le t \le 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

The initial conditions are the same as before.

(a) (MP) Solve the resulting system for the displacement x in the region $t \in [0, 2\pi]$.

(b) (MP) Show that

$$x(2\pi) \approx 0.0419, \qquad \dot{x}(2\pi) \approx 1.2425.$$

(c) (BH) Why should x and $\dot{x}$ be continuous at $t = 2\pi$?

Solution. Since the change in forcing at $t = 2\pi$ affects only the acceleration, not the velocity or position, x and $\dot{x}$ should be continuous.

(d) (MP) Using your answer to (c), calculate x in the region $t > 2\pi$.

(e) (MP) Plot your solution x for $t \in [0, 4\pi]$.

6. (BH) Consider the dimensional equation for the forced spring given in class:

$$M\ddot{x} + b\dot{x} + kx = F(t). \tag{5.4}$$

If $M = 1$, $b = 1$, $k = 2$, and $F = -\sin t$, find the steady-state solution for the displacement.

Solution. Substituting the parameters into (5.4), we obtain

$$\ddot{x} + \dot{x} + 2x = -\sin t. \tag{D}$$

To find the particular solution (which is the steady state), we substitute $x_{\mathrm{p}} = c_c \cos t + c_s \sin t$ into (D) to obtain

$$\begin{aligned} -(c_c \cos t + c_s \sin t) + (c_s \cos t - c_c \sin t) + 2(c_c \cos t + c_s \sin t) &= -\sin t \\ c_c + c_s &= 0 \\ c_s - c_c &= -1 \\ c_c = \frac{1}{2}, \qquad c_s &= -\frac{1}{2}. \end{aligned}$$

But the particular solution is the steady state, so we have that the steady state is given by

$$x_{\mathrm{s}} = \frac{\cos t - \sin t}{2}.$$

7. When scaled in the proper manner, the equation for the velocity $v(r)$ in a cylindrical pipe is given by

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = -a, \quad r \in [0, 1]; \qquad v(1) = 0, \tag{5.5}$$

where $a > 0$ and r is distance from the center.

(a) (BH) Find the general solution to (5.5).

Solution. Letting $w = dv/dr$ in (5.5), we obtain

$$\frac{dw}{dr} + \frac{1}{r}w = -a,$$

which is a first-order equation with $p(r) = 1/r$:

$$\mu = \exp\left(\int \frac{dr}{r}\right) = \exp(\log r) = r,$$
$$\frac{d(rw)}{dr} = -ar$$
$$rw = -\frac{ar^2}{2} + c_1.$$

Then using our definition of w, we have

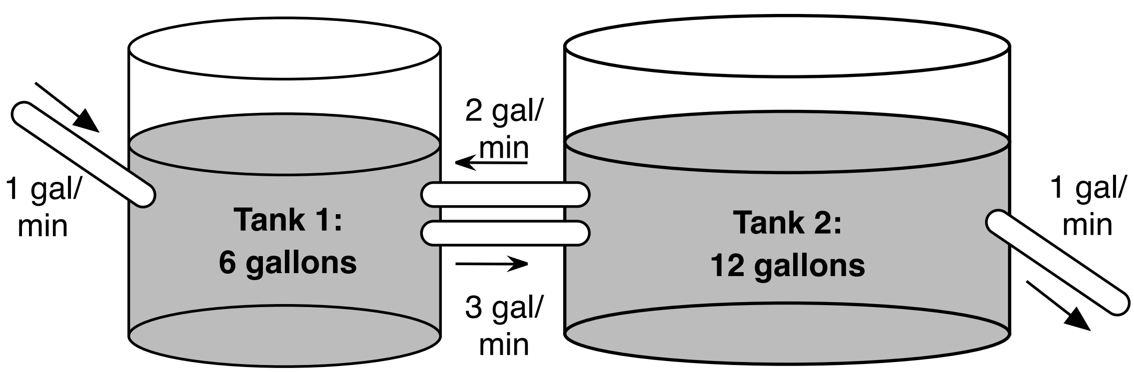
$$\frac{dv}{dr} = -\frac{ar}{2} + \frac{c_1}{r}$$
$$v = -\frac{ar^2}{4} + c_1 \log r + c_2$$
$$v(1) = -\frac{a}{4} + c_2 = 0$$
$$c_2 = \frac{a}{4}$$
$$v(r) = \frac{a(1-r^2)}{4} + c_1 \log r. \tag{E}$$

(b) (BH) Write the solution to the physical system.

Solution. We expect the velocity to remain bounded at $r = 0$, so we see that we must take $c_1 = 0$ in (E) to obtain

$$v(r) = \frac{a(1-r^2)}{4}.$$

(c) (MP) Plot your solution for $r \in [0, 1]$ and $a = 1, 2, 3, 4$.



8. Consider the system of two interconnected tanks shown above. Tank 1 contains six gallons of solution and tank 2 contains twelve gallons of a chemical solution (see figure). Fresh water is pumped into tank 1 at the rate of 1 gal/min. Solution from tank 1 to tank 2 is pumped at 3 gal/min. Solution is pumped from tank 2

to tank 1 at 2 gal/min, and solution is removed from tank 2 at 1 gal/min. Let m_i be the mass of the chemical in tank i.

(a) (BH) Will the volume of solution in the tanks ever change? Why or why not?

Solution. 3 gal/min leaves Tank 1 (going to Tank 2), but 3 gal/min comes in (1 gal/min of fresh water, and 2 gal/min from Tank 2). 3 gal/min enters Tank 2 (from Tank 1), but 3 gal/min leaves (1 gal/min to the drain, and 2 gal/min to Tank 1). So the tanks will remain at the same level.

(b) (BH) Show that the system of differential equations governing m_1 and m_2 is given by

$$\begin{aligned} 6\dot{m}_1 &= -3m_1 + m_2, && (5.6a)\\ 4\dot{m}_2 &= 2m_1 - m_2. && (5.6b) \end{aligned}$$

Solution. The change of mass in tank 1 ($\dot{m}_1$, in g/min) is given by the influx from tank 2 and the outflow to tank 2. The influx from tank 2 is given by the current concentration in tank 2 ($m_2/12$, in g/gal), times the flow rate (2 gal/min). Therefore, the influx is $m_2/6$, and the outflow is $(-m_1/6)(3) = -m_1/2$:

$$\begin{aligned} \dot{m}_1 &= -\frac{m_1}{2} + \frac{m_2}{6}\\ 6\dot{m}_1 &= -3m_1 + m_2. \end{aligned}$$

The change of mass in tank 2 ($\dot{m}_1$, in g/min) is given by the influx from tank 1 and the outflow to tank 2 and the drain. The influx from tank 1 is given by $(m_1/6)(3) = m_1/2$, while the total outflow is given by $(-m_2/12)(2+1) = -m_2/4$. Therefore, we have

$$\begin{aligned} \dot{m}_2 &= \frac{m_1}{2} - \frac{m_2}{4} && \text{(F)}\\ 4\dot{m}_2 &= 2m_1 - m_2. \end{aligned}$$

(c) (BH) Combine equations (5.6) into a single second-order equation for m_1.

Solution. Taking the derivative of (5.6a) with respect to t and substituting in (F), we have

$$\begin{aligned} 6\ddot{m}_1 &= -3\dot{m}_1 + \dot{m}_2 = -3\dot{m}_1 + \left(\frac{m_1}{2} - \frac{m_2}{4}\right) = -3\dot{m}_1 + \frac{m_1}{2} - \frac{6\dot{m}_1 + 3m_1}{4}\\ 6\ddot{m}_1 &= -\frac{9\dot{m}_1}{2} - \frac{m_1}{4}. \end{aligned}$$

(d) (MP) Solve your answer to (c) and find expressions for m_1 and m_2 if $m_1(0) = 5$, $m_2(0) = 25$.

(e) (BH) What happens to m_1 and m_2 as $t \to \infty$? Explain your result physically.

Solution. Both m_1 and m_2 go to zero because we are adding fresh water and draining solution.

9. (BH) Write the second-order linear ordinary differential equation

$$(\cos t)\ddot{w} + t^2\dot{w} + e^{-t}w = 0$$

as a system of two first-order linear ordinary differential equations.

Solution. Letting $x = \dot{w}$, we have

$$\begin{aligned} \dot{w} &= x, \\ (\cos t)\dot{x} + t^2 x + e^{-t}w &= 0. \end{aligned}$$

10. (BH) Without memory effects, Newton's Second Law for the displacement $x(t)$ of an object of mass m is given by

$$m\ddot{x} = F(x, \dot{x}, t). \tag{5.7}$$

Introduce the momentum p to transform (5.7) into a system of two coupled first-order equations.

Solution. The momentum is defined by

$$m\dot{x} = p. \tag{G.1}$$

Substituting this result into (5.7), we have

$$\dot{p} = F\left(x, \frac{p}{m}, t\right). \tag{G.2}$$

The two equations in (G) form a first-order system, as desired.

In[ ]:=
```
Quit[]
```

HW1 (Checked)

HW2 (Checked)

HW3 (Checked)

HW4 (Checked)

HW5 (Checked)

Number 1b.

In[1]:=
```
eq4 = {x''[t] + x'[t] / 32 + 96 * x[t] == 0, x[0] == 1 / 7, x'[0] == 0}
sol4 = DSolveValue[eq4, x[t], t]
```

Out[1]= $$\left\{96\,x[t] + \frac{x'[t]}{32} + x''[t] == 0,\ x[0] == \frac{1}{7},\ x'[0] == 0\right\}$$

Out[2]= $$\frac{e^{-t/64}\left(393215\,\mathrm{Cos}\left[\frac{\sqrt{393215}\,t}{64}\right] + \sqrt{393215}\,\mathrm{Sin}\left[\frac{\sqrt{393215}\,t}{64}\right]\right)}{2752505}$$

Number 1c.

First we compute the amplitude. To do this, we just compute c1 and c2, then the required square root.

In[3]:=
```
nsol4 = Expand[sol4]
nsol4[[2]]
c1 = % /. (Sin[Sqrt[393215] t/64] -> 1)
nsol4[[1]]
c2 = % /. (Cos[Sqrt[393215] t/64] -> 1)
amp = Sqrt[c1^2 + c2^2]
```

Out[3]= $\frac{1}{7} e^{-t/64} \mathrm{Cos}\left[\frac{\sqrt{393215}\, t}{64}\right] + \frac{e^{-t/64} \mathrm{Sin}\left[\frac{\sqrt{393215}\, t}{64}\right]}{7 \sqrt{393215}}$

Out[4]= $\frac{e^{-t/64} \mathrm{Sin}\left[\frac{\sqrt{393215}\, t}{64}\right]}{7 \sqrt{393215}}$

Out[5]= $\frac{e^{-t/64}}{7 \sqrt{393215}}$

Out[6]= $\frac{1}{7} e^{-t/64} \mathrm{Cos}\left[\frac{\sqrt{393215}\, t}{64}\right]$

Out[7]= $\frac{e^{-t/64}}{7}$

Out[8]= $\frac{256}{7} \sqrt{\frac{6}{393215}} \sqrt{e^{-t/32}}$

Then we set this equal to the threshold and solve for t_*:

In[9]:=
```
Solve[amp == 0.01, t]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[9]= {{t → 170.193}}

Number 3c.

In[10]:=
```
sol3 = (1 - Exp[V * x]) / (1 - Exp[V])
sol30 = x
Show[Plot[Evaluate[Table[sol3, {V, -2, 2, 4}]],
  {x, 0, 1}, PlotStyle → {Red, Green }], Plot[sol30, {x, 0, 1}]]
```

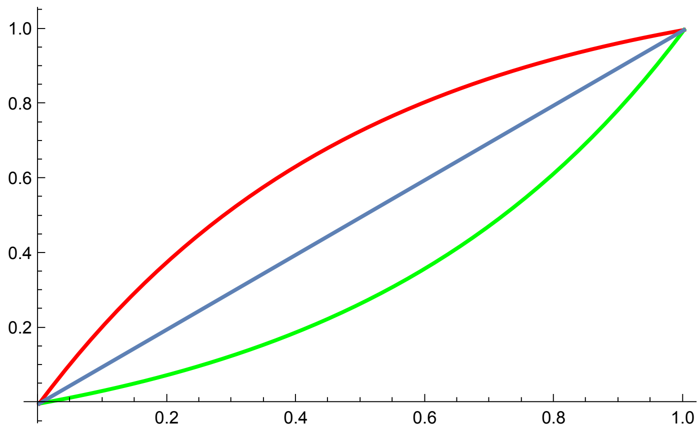
Out[10]=

$$\frac{1 - e^{V x}}{1 - e^{V}}$$

Out[11]=

x

Out[12]=



The blue case is the steady state without any velocity. If the velocity is positive (green curve), the heat provided on the right side is being being opposed by the velocity, and the heat in the whole region is less. If the velocity is negative (red curve), the heat provided on the left side is being assisted by the flow to the left, and the temperature is higher everywhere.

Number 5a.

In[13]:=
```
eq9 = x''[t] + x'[t] / 32 + 96 * x[t] == F[t]
sys9a = {eq9 /. F[t] → 4 * Sin[t], x[0] == 1 / 7, x'[0] == 0}
sol9a = DSolveValue[sys9a, x[t], t]
```

Out[13]=

$$96\, x[t] + \frac{x'[t]}{32} + x''[t] == F[t]$$

Out[14]=

$$\left\{96\, x[t] + \frac{x'[t]}{32} + x''[t] == 4\, \mathrm{Sin}[t],\ x[0] == \frac{1}{7},\ x'[0] == 0\right\}$$

Out[15]=

$$-\frac{1}{25\,437\,552\,960\,505} e^{-t/64} \left(352\,320\,640\, e^{t/64}\, \mathrm{Cos}[t] - 3\,634\,288\,457\,855\, \mathrm{Cos}\left[\frac{\sqrt{393\,215}\; t}{64}\right] - 1\,071\,054\,745\,600\, e^{t/64}\, \mathrm{Sin}[t] + 165\,083\,263\, \sqrt{393\,215}\, \mathrm{Sin}\left[\frac{\sqrt{393\,215}\; t}{64}\right]\right)$$

Number 5b.

In[16]:=
```
ic5d1 = x[2*Pi] == N[sol9a /. (t → 2*Pi)]
ic5d2 = x'[2*Pi] == N[D[sol9a, t] /. (t → 2*Pi)]
```

Out[16]=

$x[2\pi] == 0.0419376$

Out[17]=

$x'[2\pi] == 1.24252$

Number 5d.

In[18]:=
```
sys9b = {eq9 /. F[t] → 0, ic5d1, ic5d2}
sol9b = DSolveValue[sys9b, x[t], t]
```

Out[18]=

$$\left\{96\,x[t] + \frac{x'[t]}{32} + x''[t] == 0,\ x[2\pi] == 0.0419376,\ x'[2\pi] == 1.24252\right\}$$

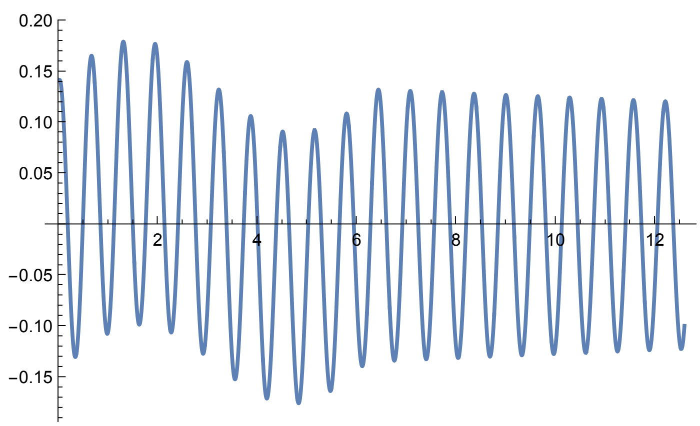
Out[19]=

$$0.147394\,e^{-t/64}\left(1.\,\mathrm{Cos}\left[\frac{\sqrt{393215}\,t}{64}\right] - 0.0179674\,\mathrm{Sin}\left[\frac{\sqrt{393215}\,t}{64}\right]\right)$$

Number 5e.

In[20]:=
```
plot9a = Plot[sol9a, {t, 0, 2*Pi}];
plot9b = Plot[sol9b, {t, 2*Pi, 4*Pi}];
Show[plot9a, plot9b, PlotRange → All]
```

Out[22]=



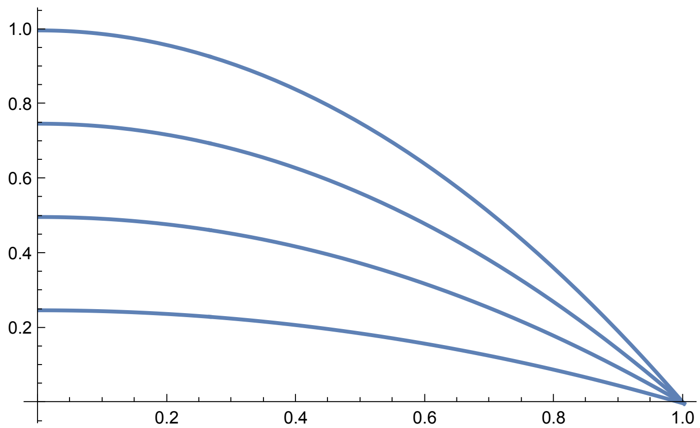
Number 7c.

In[23]:=
```
eq7 = a * (1 - r^2) / 4
plot7c = Plot[Table[eq7, {a, 1, 4}], {r, 0, 1}]
```

Out[23]=

$$\frac{1}{4} a \left(1 - r^2\right)$$

Out[24]=



Number 8d.

First we set up the proper system. Note that we use `ToRules` to convert the equation into a substitution rule to figure out $\dot{m}_1(0)$. We could just have easily set up two definitions.

In[25]:=
```
eq1a = 6 * m1 '[t] == -3 * m1[t] + m2[t]
eq1c = 6 * m1 ''[t] == -9 * m1 '[t] / 2 - m1[t] / 4
ic1 = m1[0] == 5
icforsub = ToRules[ic1][[1]]
ic2 = m2[0] → 25
(eq1a /. t → 0) /. {icforsub, ic2}
ic3 = Roots[%, m1′[0]]
```

Out[25]=
$$6\, m1'[t] == -3\, m1[t] + m2[t]$$

Out[26]=
$$6\, m1''[t] == -\frac{m1[t]}{4} - \frac{9\, m1'[t]}{2}$$

Out[27]=
$$m1[0] == 5$$

Out[28]=
$$m1[0] \to 5$$

Out[29]=
$$m2[0] \to 25$$

Out[30]=
$$6\, m1'[0] == 10$$

Out[31]=
$$m1'[0] == \frac{5}{3}$$

In[32]:=
```
m1sol = DSolveValue[{eq1c, ic1, ic3}, m1[t], t]
Solve[eq1a /. {m1[t] → m1sol, m1 '[t] → D[m1sol, t]}, m2[t]]
```

Out[32]=
$$-\frac{5}{114}\left(-57\, e^{\left(-\frac{3}{8}-\frac{\sqrt{\frac{19}{3}}}{8}\right)t} + 17\sqrt{57}\, e^{\left(-\frac{3}{8}-\frac{\sqrt{\frac{19}{3}}}{8}\right)t} - 57\, e^{\left(-\frac{3}{8}+\frac{\sqrt{\frac{19}{3}}}{8}\right)t} - 17\sqrt{57}\, e^{\left(-\frac{3}{8}+\frac{\sqrt{\frac{19}{3}}}{8}\right)t}\right)$$

Out[33]=
$$\left\{\left\{m2[t] \to -\frac{5}{38}\left(-95\, e^{\left(-\frac{3}{8}-\frac{\sqrt{\frac{19}{3}}}{8}\right)t} + 9\sqrt{57}\, e^{\left(-\frac{3}{8}-\frac{\sqrt{\frac{19}{3}}}{8}\right)t} - 95\, e^{\left(-\frac{3}{8}+\frac{\sqrt{\frac{19}{3}}}{8}\right)t} - 9\sqrt{57}\, e^{\left(-\frac{3}{8}+\frac{\sqrt{\frac{19}{3}}}{8}\right)t}\right)\right\}\right\}$$

HW6 (Checked)

HW7 (Checked)

HW8 (Checked)