Homework Set 3 Solutions

1. (BH) For the equation
$$\ddot{y} + 5\dot{y} + 6y = 0,$$
find the fundamental set $\{y_1(t), y_2(t)\}$ where
$$y_1(0) = 1, \quad \dot{y}_1(0) = 0; \qquad y_2(0) = 0, \quad \dot{y}_2(0) = 1.$$

Solution. Substituting $y = e^{\lambda t}$, we obtain
$$\begin{aligned} \lambda^2 + 5\lambda + 6 &= 0 \\ (\lambda + 3)(\lambda + 2) &= 0, \end{aligned}$$
so solutions are of the form $y = c_1 e^{-3t} + c_2 e^{-2t}$. Therefore, for y_1 we must solve
$$\begin{aligned} y_1(0) = c_1 + c_2 &= 1 \\ \dot{y}_1(0) = -3c_1 - 2c_2 &= 0. \end{aligned} \qquad \Longrightarrow \qquad c_1 = -2$$
$$y_1(t) = -2e^{-3t} + 3e^{-2t},$$
and for y_2 we must solve
$$\begin{aligned} y_2(0) = c_1 + c_2 &= 0 \\ \dot{y}_2(0) = -3c_1 - 2c_2 &= 1. \end{aligned} \qquad \Longrightarrow \qquad c_1 = -1, \quad c_2 = 1$$
$$y_2(t) = -e^{-3t} + e^{-2t}.$$

2. (BH) Consider the equation
$$(t^2 - 4t + 3)\ddot{y} + 3t\dot{y} + \frac{3y}{\log t} = 0.$$
Find all intervals where this equation is guaranteed to have a unique solution.

Solution. Rewriting the equation in standard form, we have
$$\ddot{y} + \frac{3t}{t^2 - 4t + 3}\dot{y} + \frac{3y}{(t-3)(t-1)\log t} = 0,$$
where we have used the fact that $t^2 - 4t + 3 = (t-3)(t-1)$. The coefficient of y is undefined whenever $t = 1$, $t = 3$, and $t < 0$. Therefore, the equation has a unique solution in any interval *not* containing those points.

3. Consider the equation

$$(t-5)\ddot{y} - t^2\dot{y} - y\sqrt{7-t} = 0, \quad t > 0; \qquad y(0) = 1, \quad \dot{y}(0) = 0. \tag{3.1}$$

(a) (BH) Find the interval $[a, b)$ where (3.1) is guaranteed to have a unique solution.

Solution. Rewriting the equation in standard form, we have

$$\ddot{y} - \frac{t^2}{t-5}\dot{y} - \frac{\sqrt{7-t}}{t-5}y = 0.$$

The coefficient of y is undefined whenever $t = 5$ or $t > 7$. Since the initial conditions are given at $t = 0$, the equation has a unique solution when $t \in [0, 5)$.

(b) (MP) Plot the solution to (3.1) in $[a, b]$. What happens?

(c) (MP) Plot the solution to (3.1) in $[a, c]$, where $c > b$. What happens?

4. (BH) Prove that if $\dot{y}_1$ and $\dot{y}_2$ are zero at the same point in I, they cannot be a fundamental set of solutions on that interval.

Solution. Let t_0 be the point at which $\dot{y}_1(t_0) = \dot{y}_2(t_0) = 0$. But then the Wronskian at t_0 is given by

$$W(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ 0 & 0 \end{vmatrix} = 0.$$

But by the theorem we know that $W \neq 0$ in an interval for the two solutions to be a fundamental set. Hence the result is proved.

5. (BH) By considering their Wronskian, show that

$$y_1(\theta) = e^{a\theta}\cos b\theta, \quad y_2(\theta) = e^{a\theta}\sin b\theta, \qquad b \neq 0,$$

are linearly independent for all θ.

Solution. Calculating the Wronskian, we have

$$\begin{aligned} W &= \begin{vmatrix} e^{a\theta}\cos b\theta & e^{a\theta}\sin b\theta \\ e^{a\theta}(a\cos b\theta - b\sin b\theta) & e^{a\theta}(a\sin b\theta + b\cos b\theta) \end{vmatrix} \\ &= e^{2a\theta}[\cos b\theta(a\sin b\theta + b\cos b\theta) - \sin b\theta(a\cos b\theta - b\sin b\theta)] \\ &= e^{2a\theta}[b(\cos^2 b\theta + \sin^2 b\theta)] = be^{2a\theta} \neq 0, \end{aligned}$$

where we have used the fact that $b \neq 0$. Hence $W \neq 0$, and the functions are linearly independent.

6. Consider the equation and boundary condition

$$\frac{d}{dr}\left(r^{1/2}\frac{dy}{dr}\right) = 0, \tag{3.2a}$$

$$y(0) = 1. \tag{3.2b}$$

(a) (BH) Solve the system (3.2) for y. (You should obtain a one-dimensional family of solutions.)

Solution. Integrating (3.2a), we have

$$\begin{aligned} r^{1/2}\frac{dy}{dr} &= C_1 \\ \frac{dy}{dr} &= C_1 r^{-1/2} \qquad \text{(A.1)} \\ y &= 2C_1 r^{1/2} + C_2 \\ y(0) = C_2 &= 1 \\ y(t) &= 2C_1 r^{1/2} + 1. \qquad \text{(A.2)} \end{aligned}$$

(b) (MP) Plot various integral curves of your solution to (a) for $r \in [0, 1]$. What happens near $r = 0$?

(c) (BH) Does a solution to (3.2a) with the boundary conditions

$$y(0) = 1, \qquad \frac{dy}{dr}(0) = 1$$

exist? Discuss your answer in light of Theorem 3.2.1.

Solution. Taking the derivative of (A.2), we have

$$\frac{dy}{dr} = C_1 r^{-1/2},$$

as given by (A.1). Thus every solution has a derivative which is undefined at $r = 0$. Rewriting (3.2a) in the theoretical form, we have

$$\begin{aligned} r^{1/2}\frac{d^2y}{dr^2} + \frac{1}{2r^{1/2}}\frac{dy}{dr} &= 0 \\ \frac{d^2y}{dr^2} + \frac{1}{2r}\frac{dy}{dr} &= 0. \end{aligned}$$

Since the coefficient of dy/dr doesn't exist at $r = 0$, we have no guarantee of a unique solution.

7. Consider the equation

$$t^2 y^{(4)} + \sqrt{3-t}\, y^{(3)} + \log(t-1)\ddot{y} + \sin t\dot{y} + 3y = 0. \qquad (3.3)$$

Find all intervals where (3.3) has a unique solution.

Solution. Rewriting (3.3) in the standard theoretical form, we have

$$y^{(4)} + \frac{\sqrt{3-t}}{t^2} y^{(3)} + \frac{\log(t-1)}{t^2}\ddot{y} + \frac{\sin t}{t^2}\dot{y} + \frac{3}{t^2} y = 0.$$

Therefore, discontinuities occur at $t = 0$ (from each term), $t > 3$ (from the second term), and $t \leq 1$ (from the third term). Therefore, the only interval where (3.3) has a unique solution is $(1, 3]$.

8. Consider the initial-value problem

$$4y^{(3)} + 4\ddot{y} - 3\dot{y} = 0, \qquad y(0) = 3, \quad \dot{y}(0) = -2, \quad \ddot{y}(0) = 2. \tag{3.4}$$

(a) (BH) Find the solution of (3.4).

Solution. Substituting $y = e^{\lambda t}$, we have

$$4\lambda^3 + 4\lambda^2 - 3\lambda = \lambda(4\lambda^2 + 4\lambda - 3) = \lambda(2\lambda + 3)(2\lambda - 1) = 0$$

$$y(t) = c_1 + c_2 e^{-3t/2} + c_3 e^{t/2}$$

$$\left.\begin{aligned} \dot{y}(0) &= \frac{-3c_2 + c_3}{2} = -2 \\ \ddot{y}(0) &= \frac{9c_2 + c_3}{4} = 2 \end{aligned}\right. \quad \Longrightarrow \quad c_3 = -2\frac{3}{2} + 2 = -1, \quad c_2 = 1$$

$$y(0) = c_1 + c_2 + c_3 = 3 \qquad \Longrightarrow \qquad c_1 = 3$$

$$y(t) = 3 + e^{-3t/2} - e^{t/2}.$$

(b) (MP) Plot your solution for $t \in [0, 3]$.

9. Find the general solution of

$$y^{(4)} - 5\ddot{y} + 4y = 0.$$

Solution. Substituting $y = e^{\lambda t}$, we have

$$\lambda^4 - 5\lambda^2 + 4 = (\lambda^2 - 4)(\lambda^2 - 1) = (\lambda + 2)(\lambda - 2)(\lambda + 1)(\lambda - 1) = 0$$

$$y(t) = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^{-t} + c_4 e^{t}.$$

10. (MP) Plot the solution to

$$y^{(3)} - (2 + \sin t)y = 0, \quad y(0) = a, \quad \dot{y}(0) = 0, \quad \ddot{y}(0) = a^2,$$

for $t \in [0, 2\pi]$ and $a = -2, -1$, 0, 1, and 2.

In[ ]:=

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Quit[]
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HW1 (Checked)

HW2 (Checked)

HW3 (Checked)

Number 3b.

In[ ]:=

```
eq33 = {(t - 5) * y''[t] - t^2 * y'[t] - y[t] * Sqrt[7 - t] == 0, y[0] == 1, y'[0] == 0}
eq3bsolve = NDSolve[eq33, y, {t, 0, 5}];
Plot[Evaluate[y[t] /. eq3bsolve], {t, 0, 5}]
```

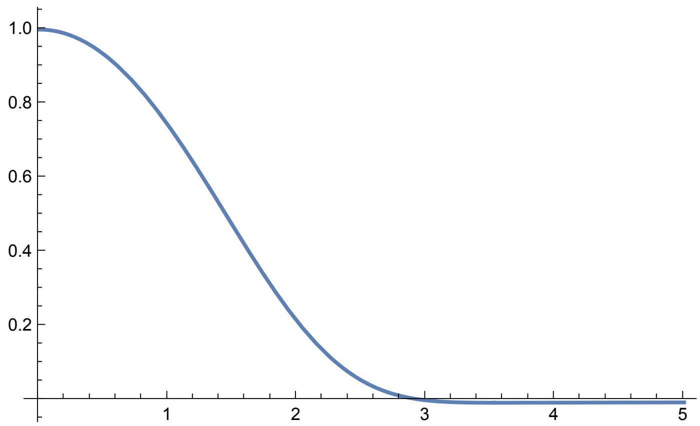
Out[ ]=

$$\left\{-\sqrt{7-t}\; y[t] - t^2 y'[t] + (-5+t)\, y''[t] == 0,\; y[0] == 1,\; y'[0] == 0\right\}$$

Power: Infinite expression $\frac{1}{0.}$ encountered.

NDSolve: The function value {0.000366174, ComplexInfinity} is not a list of numbers with dimensions {2} at {t, y[t], y′[t]} = {5., −0.0064732, 0.000366174}.

Out[ ]=

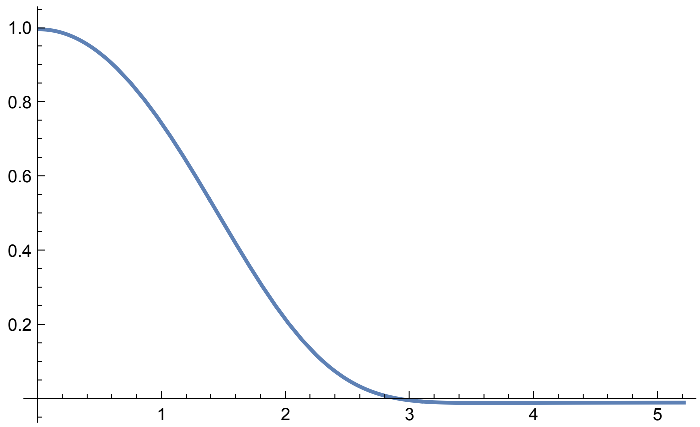


In this case, since t=5 is an endpoint of the computation, Mathematica realizes there is an issue.

Number 3c.

In[ ]:=
```
eq3csolve = NDSolve[eq33, y, {t, 0, 5.2}];
Plot[Evaluate[y[t] /. eq3csolve], {t, 0, 5.2}]
```

Out[ ]=



In this case, since t=5 is not an endpoint of the computation, Mathematica skips over it and doesn't realize there's an issue.

Number 6b.

In[ ]:=
```
sol10 = 1 + 2 * c1 * r ^ (1 / 2)
tab10 = Table[sol10, {c1, -2, 2}]
```

Out[ ]=

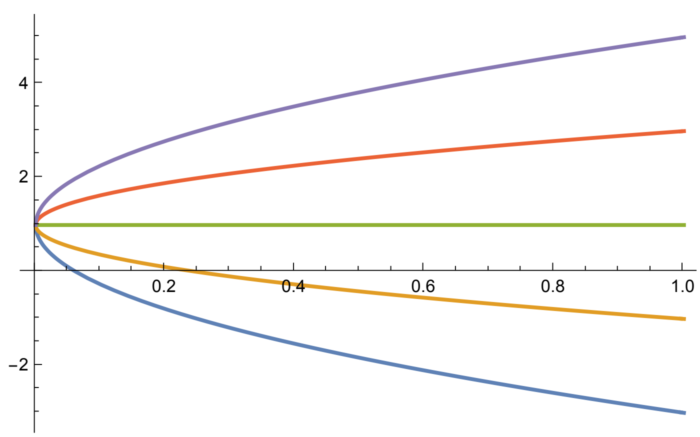
$1 + 2\,c1\,\sqrt{r}$

Out[ ]=

$\{1 - 4\sqrt{r},\ 1 - 2\sqrt{r},\ 1,\ 1 + 2\sqrt{r},\ 1 + 4\sqrt{r}\}$

In[ ]:=
```
Plot[tab10, {r, 0, 1}]
```

Out[ ]=

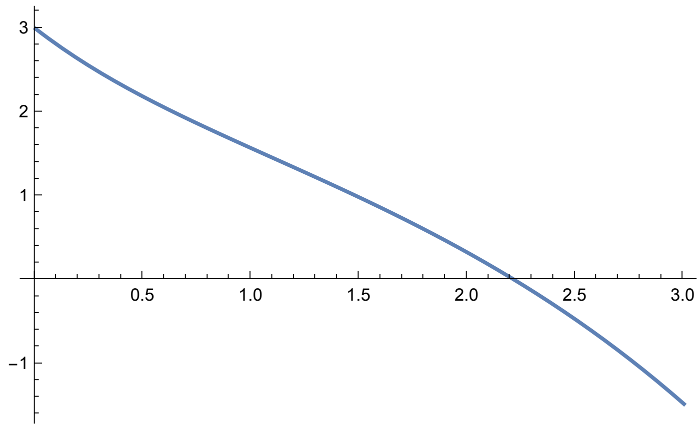


Though it's not easy to discern, all the curves come in with infinite slope, except for the equilibrium solution $y = 1$.

Number 8b.

In[2]:=
```
Plot[3 + Exp[-3 * t / 2] - Exp[t / 2], {t, 0, 3}]
```

Out[2]=



Number 10.

In[18]:=
```
eq10 = { D[y[t], {t, 3}] - (2 + Sin[t]) * y[t] == 0, y[0] == a, y'[0] == 0, y''[0] == a^2}
Table[NDSolve[eq10, y, {t, 0, 2 * Pi}], {a, -2, 2}]
Plot[Evaluate[y[t] /. %], {t, 0, 2 * Pi}]
```

Out[18]=

```
{-((2 + Sin[t]) y[t]) + y^(3)[t] == 0, y[0] == a, y′[0] == 0, y″[0] == a^2}
```

Out[19]=

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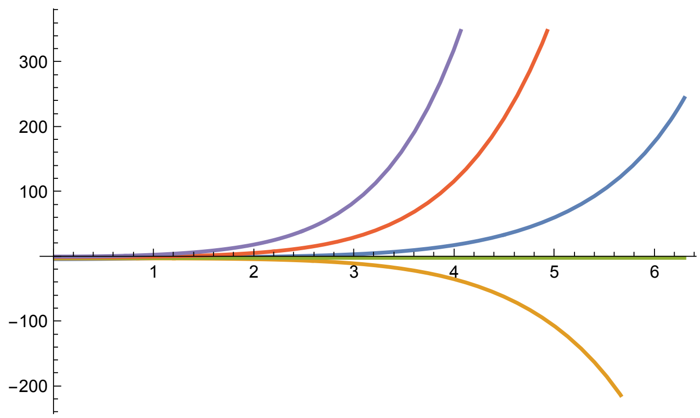
{{{y → InterpolatingFunction[ Domain: {{0., 6.28}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 6.28}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 6.28}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 6.28}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 6.28}} Output: scalar ]}}}
```

Out[20]=



HW4 (Checked)

HW5 (Checked)

HW6 (Checked)