MATH 302-010 Ordinary Differential Equations
Prof. D. A. Edwards Due: Sept. 20, 2024

Homework Set 2 Solutions

1. (BH) The cruise ship *Norwegian Joy* leaves Los Angeles with 3200 passengers, 5 of whom have the flu. By the end of the next day, 10 people have the flu.

   (a) If the number of people with the flu $N(t)$ spreads according to the exponential growth model, calculate $N(t)$.

Solution. Let $N(t)$ be the number of people with the disease at time t. Then we have

$$\begin{aligned}
\frac{dN}{dt} &= kN, \qquad N(0) = 5, \quad N(1) = 10, \\
N &= 5e^{kt}, \\
N(1) = 10 &= 5e^{k} \\
e^{k} &= 2 \\
N(t) &= 5(2)^{t}.
\end{aligned}$$

Alternatively, one can make the calculation by noting that the number of people with the disease doubles each day, starting with 5 people, so $N(t) = 5(2)^t$.

   (b) How many people will have caught the flu by the end of the week-long cruise?

Solution. Substituting in $t = 7$, we have

$$N(7) = 5(2)^7 = 5(128) = 640.$$

Therefore, 640 people will have caught the flu by the end of the cruise.

2. Suppose that due to weather and other habitat-related conditions, the growth rate actually fluctuates over the course of a year. This can be modeled by

$$\dot{N} = (r\cos kt)N, \qquad N(0) = 1,$$

   where $r > 0$ and k are constants.

   (a) (BH) Find the solution to this problem.

Solution. Separating the equation, we have

$$\begin{aligned}
\frac{dN}{N} &= r\cos kt\, dt \\
\log N &= \frac{r\sin kt}{k} + C, \\
N &= e^{C}\exp\left(\frac{r\sin kt}{k}\right) \\
N(0) = 1 &= e^{C} \\
N(t) &= \exp\left(\frac{r\sin kt}{k}\right).
\end{aligned}$$

(b) (MP) Graph your solution for $r = 2$, $k = \pi$, $t \in [0, 10]$.

3. (BH) A population $P(t)$ satisfies the following *Gompertz model*:

$$\dot{P} = rP \log\left(\frac{K}{P}\right), \qquad P(0) = 1, \tag{2.1}$$

where $r > 0$ and $K > 0$ are constants.

(a) Find the solution to this problem.

Solution. Separating (2.1), we have

$$\frac{dP}{P(\log K - \log P)} = r\,dt.$$

We may simplify the right-hand side by letting $u = \log P$, which means that $du = dP/P$:

$$\begin{aligned} \int \frac{du}{\log K - u} &= rt + C \\ -\log(\log K - u) &= rt + C \\ \log K - u = \log K - \log P &= e^{-(C+rt)}. \end{aligned}$$

We find the constant C by plugging in $t = 0$:

$$\begin{aligned} \log K - \log(P(0)) = \log K - \log 1 = \log K &= e^{-C} \\ \log\left(\frac{K}{P}\right) &= (\log K)e^{-rt} \\ \frac{K}{P} &= \exp\left((\log K)e^{-rt}\right) = K^{e^{-rt}} \\ P(t) &= K^{1-e^{-rt}}. \end{aligned} \tag{A}$$

(b) How does the solution behave as $t \to \infty$?

Solution. Taking the limit of (A) as $t \to \infty$, we have

$$P(\infty) = K^{1-0} = K.$$

(c) Are there any values of K for which the population remains constant?

Solution. Mathematically, we see that if we take $K = 1$ in (A), the population is always exactly 1. Alternatively, we see from (2.1) that $\dot{P} = 0$ if $P = K$, and if $K = 1$, $P(0) = K$.

4. Consider the equation

$$\frac{dy}{dt} = \sin(100ty), \qquad y(0) = 1.$$

(a) (BH) Explain why $y(1) < 2$.

Solution. Since the sine is always less than 1, we see that $\dot{y} < 1$ everywhere. Thus $y(t) < y(0) + t$ everywhere. Since $y(0) = 1$, $y(1) < 2$.

(b) (BH) Write d^2y/dt^2 as a function of y and t.

Solution.

$$\begin{aligned}\frac{d^2y}{dt^2} &= 100\frac{d(ty)}{dt}\cos(100ty) = 100\left(y + t\frac{dy}{dt}\right)\cos(100ty) \\ &= 100\,[y + t\sin(100ty)]\cos(100ty), \qquad \text{(B)}\end{aligned}$$

where we have used the problem statement.

(c) (BH) Let $t \in [0, 1]$. Explain why if we take $\Delta t \approx 1/150$, the error term in Euler's method is roughly the same size as the Euler step itself. This then enforces an upper bound on h.

Solution. The terms in the Taylor expansion are

$$y(t + \Delta t) = y(t) + \Delta t\frac{dy}{dt}(t) + \frac{(\Delta t)^2}{2}\frac{d^2y}{dt^2}(t_*),$$

where t_* is some point in $[t, t + \Delta t]$. The error and the time step are the same size when

$$\left|\Delta t\frac{dy}{dt}(t)\right| = \left|\frac{(\Delta t)^2}{2}\frac{d^2y}{dt^2}(t_*)\right|.$$

We know that dy/dt is no larger than 1, and since y is no larger than 2, by (B) we have that d^2y/dt^2 is no larger than $100(2 + 1)(1) = 300$. So we have

$$\begin{aligned}\Delta t &= 150(\Delta t)^2. \\ \Delta t &= \frac{1}{150}.\end{aligned}$$

(d) (MP) Plot the solution for $t \in [0, 1]$ using the default Mathematica solver, Euler's method with $\Delta t = 1/50$, and Euler's method with $\Delta t = 1/450$. Compare your results.

5. (MP) Consider the equation

$$\dot{y} = t^2(y^2 - 1), \qquad y(0) = 0. \qquad (2.2)$$

(a) Calculate the solution.

(b) Let $E(t; h)$ be the solution of (2.2) calculated in Mathematica using Euler's method with stepsize Δt. Calculate and store the ordered pairs

$$[\Delta t, |E(1; \Delta t) - y(1)|], \qquad \Delta t = \frac{1}{N}, \quad N = 10, 11, \ldots, 50,$$

where $y(1)$ is the exact solution at $t = 1$.

(c) Plot your answer to (b) and verify that the points lie on a line. Thus we know that the error is proportional to Δt, as predicted by the theory.

6. (BH) Consider the differential equation

$$2\ddot{y} + 3\dot{y} + y = 0.$$

(a) Find the general solution. Describe the long-time behavior.

Solution. Substituting $y = e^{\lambda t}$, we obtain

$$2\lambda^2 + 3\lambda + 1 = (2\lambda + 1)(\lambda + 1) = 0$$
$$y(t) = c_1 e^{-t/2} + c_2 e^{-t}. \tag{C}$$

Therefore, $y \to 0$ as $t \to \infty$.

(b) Calculate the specific solution for $y(0) = 3$, $\dot{y}(0) = -4$.

Solution. Substituting (C) into our boundary conditions, we have

$$\begin{aligned} y(0) &= c_1 + c_2 = 3 \\ \dot{y}(0) &= -\frac{c_1}{2} - c_2 = -4 \end{aligned} \quad\Longrightarrow\quad \frac{c_1}{2} = -1 \quad\Longrightarrow\quad c_1 = -2, \quad c_2 = 5.$$

$$y(t) = 5e^{-t} - 2e^{-t/2}.$$

7. (BH) Write down *all* equations of the form $a\ddot{y} + b\dot{y} + cy = 0$ such that the solution y approaches a multiple of e^{2t} as $t \to \infty$.

Solution. Substituting $y = e^{\lambda t}$, we obtain $a\lambda^2 + b\lambda + c = 0$. If the solution approaches a multiple of e^{2t}, we see that the quadratic equation must have $\lambda = 2$ as a root and another root λ_2 which is *less than* $\lambda = 2$. (Otherwise, the solution would approach $e^{\lambda_2 t}$.) So we must have

$$a(\lambda - 2)(\lambda - \lambda_2) = 0$$
$$a\lambda^2 - a(2 + \lambda_2)\lambda + 2a\lambda_2 = 0, \qquad \lambda_2 < 2.$$

8. (BH) Consider the following system of coupled first-order ODEs:

$$\dot{x} = 4x + 2y, \tag{2.3a}$$
$$\dot{y} = 3x - y. \tag{2.3b}$$

(a) Eliminate x from the system to obtain a second-order ODE for y.

Solution. Substituting (2.3a) into the derivative of (2.3b), we obtain

$$\ddot{y} = 3\dot{x} - \dot{y} = 3(4x + 2y) - \dot{y} = 12x + 6y - \dot{y}.$$

Solving (2.3b) for $12x$, we have

$$12x = 4\dot{y} + 4y. \tag{D}$$

Substituting (D) into our ODE, we have

$$\begin{aligned}\ddot{y} - 6y + \dot{y} &= 4\dot{y} + 4y \\ \ddot{y} - 3\dot{y} - 10y &= 0.\end{aligned}$$

(b) Show that the general solution for y is

$$y(t) = c_1 e^{-2t} + c_2 e^{5t},$$

and find the corresponding general solution for x.

Solution. Substituting $y = e^{\lambda t}$, we obtain

$$\begin{aligned}\lambda^2 - 3\lambda - 10 &= 0 \\ (\lambda + 2)(\lambda - 5) &= 0 \\ y(t) &= c_1 e^{-2t} + c_2 e^{5t}.\end{aligned}$$

Then using this result in (D), we obtain

$$\begin{aligned}12x = 4(\dot{y} + y) &= 4\left[(-2c_1 e^{-2t} + 5c_2 e^{5t}) + (c_1 e^{-2t} + c_2 e^{5t})\right] \\ 12x &= -4c_1 e^{-2t} + 24c_2 e^{5t} \\ x &= -\frac{c_1}{3} e^{-2t} + 2c_2 e^{5t}.\end{aligned}$$

9. (MP) Consider the differential equation

$$\ddot{y} + 3\dot{y} - 12y = 0, \quad y(0) = y_0, \quad \dot{y}(0) = 0. \tag{2.4}$$

(a) Using `DSolve`, calculate the solution of (2.4).

(b) Plot your results for $y_0 = -2, -1, 0, 1, 2$, and $t \in [0, 1/2]$.

10. Consider the differential equation

$$\ddot{y} + 2\dot{y} + (1 - \epsilon)y = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1 - \epsilon. \tag{2.5}$$

(a) (MP) Use `NDSolve` to plot the solution to (2.5) for $\epsilon = -0.2, -0.1, 0, 0.1$, and 0.2, and $t \in [0, 2]$.

(b) (BH) Is your graph for $\epsilon = 0$ consistent with what you would expect from substituting in $e^{\lambda t}$? (We will resolve this paradox in a later section.)

Solution. When $\epsilon = 0$, (2.5) becomes

$$\ddot{y} + 2\dot{y} + y = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1.$$

Substituting in $y = e^{\lambda t}$, we have

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \qquad \Longrightarrow \qquad y = c_1 e^{-t}.$$

But we have only one solution, and we have two initial conditions. So we will have a problem solving for the constants. Moreover, this exponential solution is monotonic, and our graph has a maximum.

In[ ]:=
```
Quit[]
```

HW1 (Checked)

HW2 (Checked)

Number 2b.

In[ ]:=
```
sol10 = Exp[r * Sin[k * t] / k]
plot10 = sol10 /. {r → 2, k → Pi}
Plot[plot10, {t, 0, 10}]
```

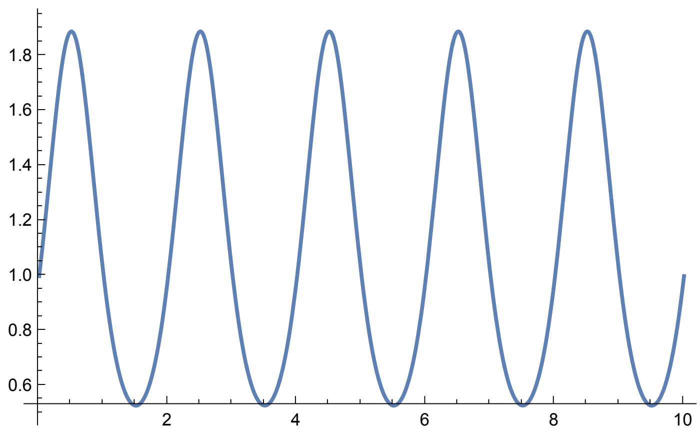
Out[ ]=

$$e^{\frac{r \operatorname{Sin}[k\, t]}{k}}$$

Out[ ]=

$$e^{\frac{2 \operatorname{Sin}[\pi\, t]}{\pi}}$$

Out[ ]=



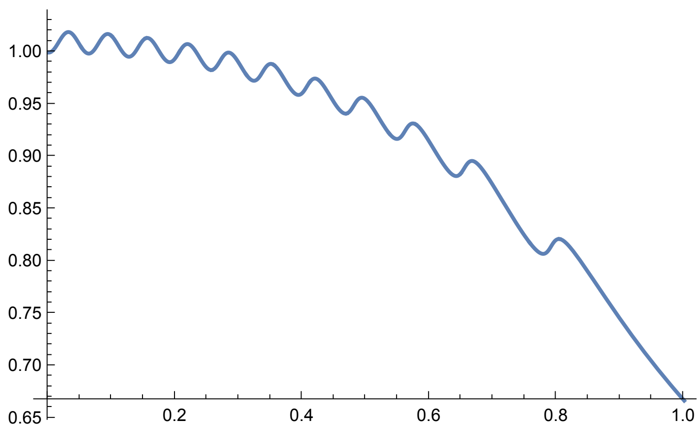
Number 4d.

Here is the equation and the graph using the default Mathematica solver.

In[∘]:=
```
eq24 = {y'[t] == Sin[100*t*y[t]], y[0] == 1}
defsolve = NDSolve[eq24, y, {t, 0, 1}];
Plot[Evaluate[y[t] /. defsolve], {t, 0, 1}]
```

Out[∘]=
```
{y′[t] == Sin[100 t y[t]], y[0] == 1}
```

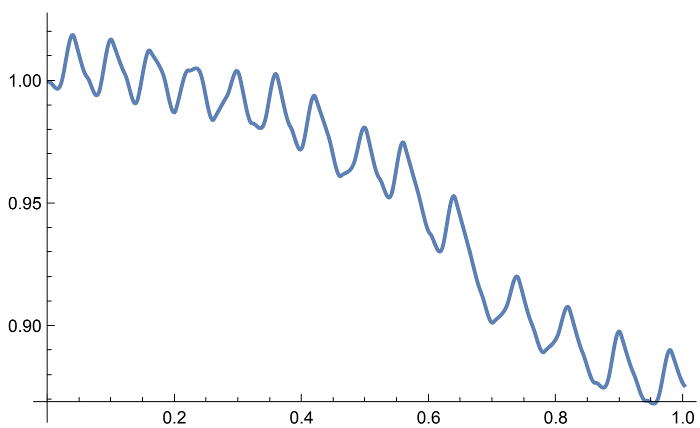
Out[∘]=



Here is the graph using Euler's method with h=1/50. This is larger than the upper bound from part (c), and so we expect bad results.

In[∘]:=
```
eulsolve =
  NDSolve[eq24, y, {t, 0, 1}, StartingStepSize → 1/50, Method → "ExplicitEuler"];
Plot[Evaluate[y[t] /. eulsolve], {t, 0, 1}]
```

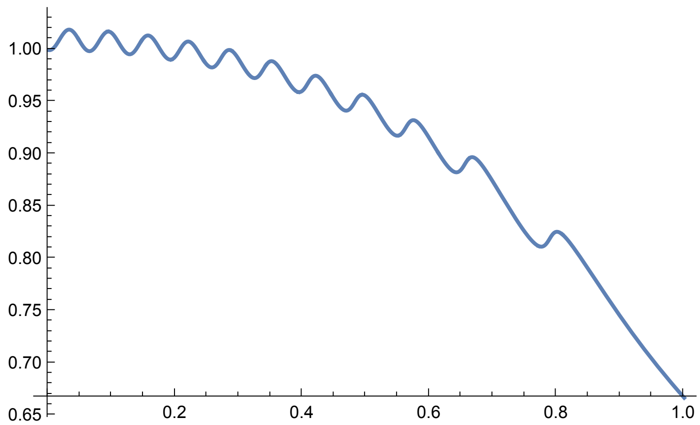
Out[∘]=



Here is the graph using Euler's method with h=1/450. This is smaller than the upper bound from part (c), and so we expect good results.

In[ ]:=
```
eulsolve2 =
  NDSolve[eq24, y, {t, 0, 1}, StartingStepSize → 1 / 450, Method → "ExplicitEuler"];
Plot[Evaluate[y[t] /. eulsolve2], {t, 0, 1}]
```

Out[ ]=



Number 5a.

Here is the equation and solution.

In[ ]:=
```
eq25 = {y'[t] == t^2 * (y[t]^2 - 1), y[0] == 0}
sol25 = DSolve[eq25, y[t], t]
```

Out[ ]=

$$\{y'[t] == t^2 (-1 + y[t]^2), y[0] == 0\}$$

DSolve: Inverse functions are being used by DSolve, so some solutions may not be found.

Out[ ]=

$$\left\{\left\{y[t] \to -\frac{-1 + e^{\frac{2t^3}{3}}}{1 + e^{\frac{2t^3}{3}}}\right\}\right\}$$

Number 5b.

Here is the exact solution at t=1, as needed. Note that we have to extract the proper part to get just the value.

In[ ]:=
```
exact25 = ( sol25[[1, 1, 2]] /. (t → 1))
```

Out[ ]=

$$-\frac{-1 + e^{2/3}}{1 + e^{2/3}}$$

Now we construct a table with all the proper NDSolve computations:

In[ ]:=
```
eul5solve = Table[NDSolve[eq25, y, {t, 0, 1},
    StartingStepSize → 1 / N, Method → "ExplicitEuler"], {N, 10, 50}];
```

Then we evaluate to get the y-values to plot:

```
In[•]:= mid5 = (y[1] /. eul5solve) - exact25
Out[•]=
{{0.0416089}, {0.0378057}, {0.0346373}, {0.0319574}, {0.0296614}, {0.0276725},
 {0.0259331}, {0.0243991}, {0.0230361}, {0.0218172}, {0.0207206}, {0.0197289},
 {0.0188276}, {0.0180051}, {0.0172513}, {0.0165581}, {0.0159184}, {0.0153262},
 {0.0147765}, {0.0142649}, {0.0137874}, {0.0133409}, {0.0129224}, {0.0125293},
 {0.0121595}, {0.0118108}, {0.0114815}, {0.0111702}, {0.0108752}, {0.0105954},
 {0.0103297}, {0.0100769}, {0.00983627}, {0.00960681}, {0.00938782}, {0.00917859},
 {0.00897848}, {0.0087869}, {0.00860333}, {0.00842727}, {0.00825827}}
```

Then we add on the x values by using Table again:

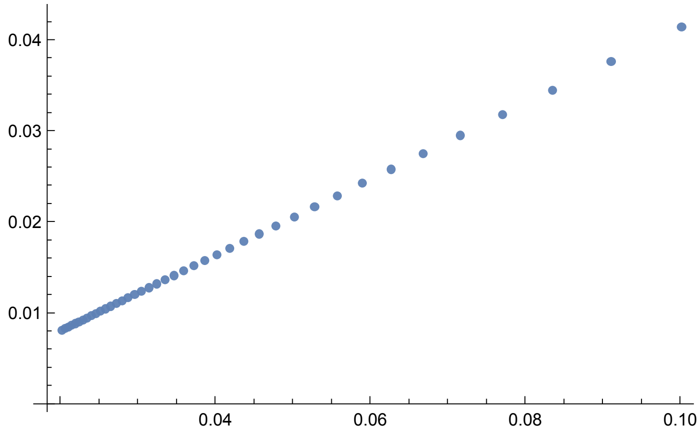
```
In[•]:= tab5 = Table[{1 / N, mid5⟦N - 9, 1⟧}, {N, 10, 50}]
Out[•]=
```

$\left\{\left\{\frac{1}{10}, 0.0416089\right\}, \left\{\frac{1}{11}, 0.0378057\right\}, \left\{\frac{1}{12}, 0.0346373\right\},\right.$
$\left\{\frac{1}{13}, 0.0319574\right\}, \left\{\frac{1}{14}, 0.0296614\right\}, \left\{\frac{1}{15}, 0.0276725\right\}, \left\{\frac{1}{16}, 0.0259331\right\},$
$\left\{\frac{1}{17}, 0.0243991\right\}, \left\{\frac{1}{18}, 0.0230361\right\}, \left\{\frac{1}{19}, 0.0218172\right\}, \left\{\frac{1}{20}, 0.0207206\right\},$
$\left\{\frac{1}{21}, 0.0197289\right\}, \left\{\frac{1}{22}, 0.0188276\right\}, \left\{\frac{1}{23}, 0.0180051\right\}, \left\{\frac{1}{24}, 0.0172513\right\},$
$\left\{\frac{1}{25}, 0.0165581\right\}, \left\{\frac{1}{26}, 0.0159184\right\}, \left\{\frac{1}{27}, 0.0153262\right\}, \left\{\frac{1}{28}, 0.0147765\right\},$
$\left\{\frac{1}{29}, 0.0142649\right\}, \left\{\frac{1}{30}, 0.0137874\right\}, \left\{\frac{1}{31}, 0.0133409\right\}, \left\{\frac{1}{32}, 0.0129224\right\},$
$\left\{\frac{1}{33}, 0.0125293\right\}, \left\{\frac{1}{34}, 0.0121595\right\}, \left\{\frac{1}{35}, 0.0118108\right\}, \left\{\frac{1}{36}, 0.0114815\right\},$
$\left\{\frac{1}{37}, 0.0111702\right\}, \left\{\frac{1}{38}, 0.0108752\right\}, \left\{\frac{1}{39}, 0.0105954\right\}, \left\{\frac{1}{40}, 0.0103297\right\},$
$\left\{\frac{1}{41}, 0.0100769\right\}, \left\{\frac{1}{42}, 0.00983627\right\}, \left\{\frac{1}{43}, 0.00960681\right\},$
$\left\{\frac{1}{44}, 0.00938782\right\}, \left\{\frac{1}{45}, 0.00917859\right\}, \left\{\frac{1}{46}, 0.00897848\right\},$
$\left.\left\{\frac{1}{47}, 0.0087869\right\}, \left\{\frac{1}{48}, 0.00860333\right\}, \left\{\frac{1}{49}, 0.00842727\right\}, \left\{\frac{1}{50}, 0.00825827\right\}\right\}$

Number 5c.

Here is a plot of the table:

In[ ]:=
```
ListPlot[tab5]
```
Out[ ]=



Note that the points all lie on a line, as predicted.

In[ ]:=

Number 9.

In[ ]:=
```
eq22 = {y''[t] + 3 * y'[t] - 12 * y[t] == 0, y[0] == y0, y'[0] == 0}
sol2 = DSolve[eq22, y[t], t]
```
Out[ ]=

$\{-12\, y[t] + 3\, y'[t] + y''[t] == 0,\ y[0] == y0,\ y'[0] == 0\}$

Out[ ]=

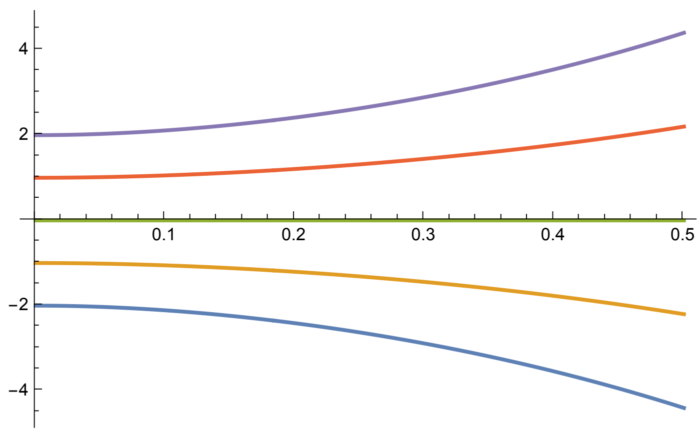
$$\left\{\left\{y[t] \to -\frac{1}{38}\left(-19\, e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right) t} + \sqrt{57}\, e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right) t} - 19\, e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right) t} - \sqrt{57}\, e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right) t}\right) y0\right\}\right\}$$

In[ ]:=
```
e2tab = Table[y[t] /. sol2, {y0, -2, 2}]
Plot[e2tab, {t, 0, 1/2}, PlotRange → All]
```

Out[ ]=

$$\left\{\left\{\frac{1}{19}\left(-19\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}+\sqrt{57}\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}-19\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}-\sqrt{57}\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}\right)\right\},\right.$$
$$\left\{\frac{1}{38}\left(-19\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}+\sqrt{57}\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}-19\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}-\sqrt{57}\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}\right)\right\},$$
$$\{0\},\left\{\frac{1}{38}\left(19\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}-\sqrt{57}\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}+19\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}+\sqrt{57}\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}\right)\right\},$$
$$\left.\left\{\frac{1}{19}\left(19\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}-\sqrt{57}\,e^{\left(-\frac{3}{2}-\frac{\sqrt{57}}{2}\right)t}+19\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}+\sqrt{57}\,e^{\left(-\frac{3}{2}+\frac{\sqrt{57}}{2}\right)t}\right)\right\}\right\}$$

Out[ ]=



Number 10a.

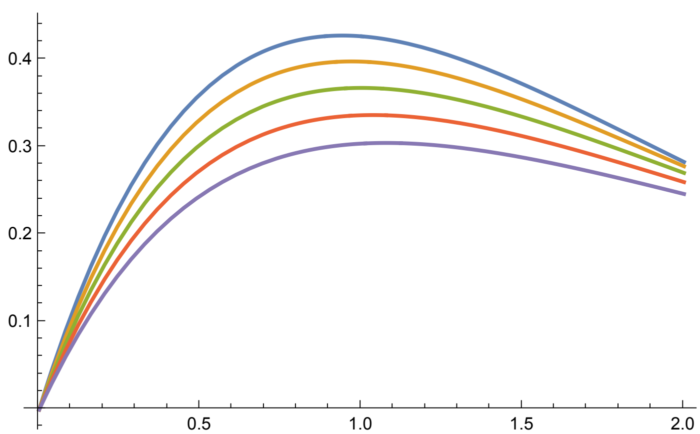
In[ ]:=
```
eq23 = {y''[t] + 2 * y'[t] + (1 - epsilon) * y[t] == 0, y[0] == 0, y'[0] == 1 - epsilon}
num5 = Table[NDSolve[eq23, y, {t, 0, 2}], {epsilon, -0.2, 0.2, 0.1}]
```

Out[ ]=
```
{(1 - epsilon) y[t] + 2 y′[t] + y″[t] == 0, y[0] == 0, y′[0] == 1 - epsilon}
```

Out[ ]=
```
{{{y → InterpolatingFunction[ Domain: {{0., 2.}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 2.}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 2.}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 2.}} Output: scalar ]}},
 {{y → InterpolatingFunction[ Domain: {{0., 2.}} Output: scalar ]}}}
```

In[ ]:=
```
Plot[Evaluate[y[t] /. num5], {t, 0, 2}]
```

Out[ ]=



HW3 (Checked)

HW4 (Checked)

HW5 (Checked)