Supplemental Study Material Solutions

1. Consider the function

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi/2, \\ 1, & \pi/2 < x < \pi. \end{cases}$$

   (a) (MP) Graph the function for $x \in [0, \pi]$.
   (b) (BH) Compute the Fourier sine series for $f(x)$. Once you have done the integrals, discuss the cases of odd and even n separately, as well as any special cases.

Solution. Here $L = \pi$, so we have

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\int_0^{\pi/2} \sin x \sin nx\, dx + \int_{\pi/2}^{\pi} \sin nx\, dx\right] \\ &= \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{\cos(1-n)x - \cos(1+n)x}{2}\, dx + \left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right\} \\ &= \frac{2}{\pi}\left\{\frac{1}{2}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi/2} - \frac{(-1)^n - \cos n\pi/2}{n}\right\} \\ &= \frac{2}{\pi}\left\{\frac{1}{2}\left[\frac{\sin(1-n)\pi/2}{1-n} - \frac{\sin(1+n)\pi/2}{1+n}\right] - \frac{(-1)^n - \cos n\pi/2}{n}\right\}. \qquad \text{(A)} \end{aligned}$$

We note that the case $n = 1$ must be handled separately. For any other odd n, the trig terms vanish, and we have

$$b_n = -\frac{2(-1)^n}{n\pi} = \frac{2}{n\pi}, \quad n > 1 \text{ odd}.$$

The special case of $n = 1$ affects only the first term, so we have

$$\begin{aligned} b_1 &= \frac{2}{\pi}\left[\int_0^{\pi/2} \sin^2 x\, dx - \frac{(-1)^1 - \cos \pi/2}{1}\right] = \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{1 - \cos 2x}{2}\, dx + 1\right] \\ &= \frac{2}{\pi}\left\{\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} + 1\right\} = \frac{2}{\pi}\left(\frac{\pi}{4} + 1\right) = \frac{1}{2} + \frac{2}{\pi}. \end{aligned}$$

If n is even, then the arguments of the sines in (A) are $n\pi$ apart, so the values are the same:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\left[\frac{\sin(1+n)\pi/2}{2}\left(\frac{1}{1-n}-\frac{1}{1+n}\right)-\frac{1-(-1)^{n/2}}{n}\right] \\
&= \frac{2}{\pi}\left[\frac{(-1)^{n/2}}{2}\left(\frac{2n}{1-n^2}\right)-\frac{1-(-1)^n}{n}\right] = \frac{2}{\pi}\left[(-1)^{n/2}\left(\frac{n}{1-n^2}+\frac{1}{n}\right)-\frac{1}{n}\right] \\
&= \frac{2}{\pi}\left[\frac{(-1)^{n/2}}{n(1-n^2)}-\frac{1}{n}\right] = \frac{2}{n\pi}\left[\frac{(-1)^{n/2}}{1-n^2}-1\right], \quad n>1 \text{ even.}
\end{aligned}$$

(c) (MP) Check your answer with Mathematica up to $n=5$.
(d) (MP) Plot your answer up to $n=15$.

2. Consider the following system:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}, \quad 0 \le x \le 1; \qquad \theta(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0. \tag{S.1}$$

(a) (BH) Separate variables, and find λ_n, $X_n(x)$, and $T_n(t)$.

Solution. Letting $\theta(x,t) = X(x)T(t)$ in (S.1), we have

$$\begin{aligned}
T'X &= X''T \\
\frac{T'}{T} &= \frac{X''}{X} = -\lambda^2 \\
X'' + \lambda^2 X &= 0, \qquad X(0) = 0, \quad X'(1) = 0, \\
X(x) &= c_1 \sin(\lambda x) + c_2 \cos(\lambda x) \\
X(0) &= c_2 = 0 \\
X'(1) &= \lambda \cos\lambda = 0 \qquad \Longrightarrow \qquad \lambda_n = (n+1/2)\pi, \\
X_n(x) &= \sin \lambda_n x. \\
T_n' &= -\lambda_n^2 T_n \\
T_n(t) &= \exp\left(-[(n+1/2)\pi]^2 t\right).
\end{aligned}$$

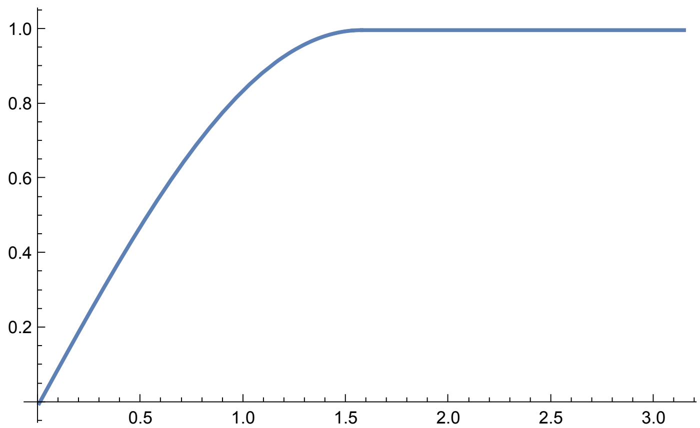
SSM (Checked)

Number 1a.

```
g1 = Sin[x] * HeavisideTheta[Pi / 2 - x] + HeavisideTheta[x - Pi / 2]
Plot[g1, {x, 0, Pi}]
```

$\text{Sin}[x]\ \text{HeavisideTheta}\left[\frac{\pi}{2} - x\right] + \text{HeavisideTheta}\left[-\frac{\pi}{2} + x\right]$

Out[ ]=



Number 1c.

```
g1 = Sin[x] * HeavisideTheta[Pi / 2 - x] + HeavisideTheta[x - Pi / 2]
FourierSinSeries[g1, x, 5]
```

$\text{Sin}[x]\ \text{HeavisideTheta}\left[\frac{\pi}{2} - x\right] + \text{HeavisideTheta}\left[-\frac{\pi}{2} + x\right]$

Out[ ]=

$$\left(\frac{1}{2} + \frac{2}{\pi}\right) \text{Sin}[x] - \frac{2\ \text{Sin}[2\ x]}{3\ \pi} + \frac{2\ \text{Sin}[3\ x]}{3\ \pi} - \frac{8\ \text{Sin}[4\ x]}{15\ \pi} + \frac{2\ \text{Sin}[5\ x]}{5\ \pi}$$

Clearly the odd cases work. For the even cases, we just check individually:

In[∘]:=
```
b1even = 2 / n / Pi * ((-1) ^ (n / 2) / (1 - n ^ 2) - 1)
b1even /. n → 2
b1even /. n → 4
```

Out[∘]=

$$\frac{2\left(-1+\frac{i^{n}}{1-n^{2}}\right)}{n\,\pi}$$

Out[∘]=

$$-\frac{2}{3\,\pi}$$

Out[∘]=

$$-\frac{8}{15\,\pi}$$

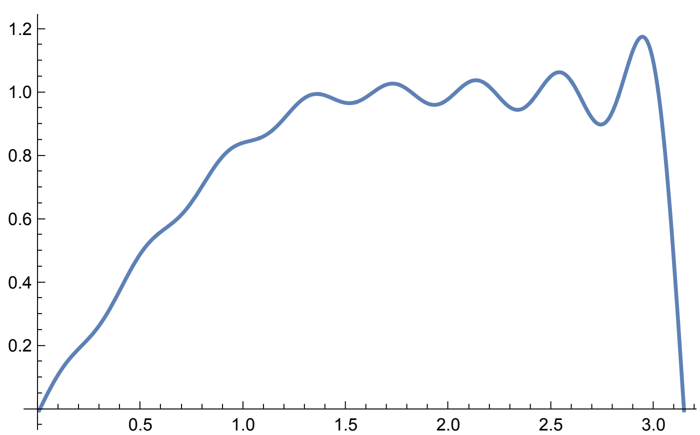
Number 1d.

In[∘]:=
```
FourierSinSeries[g1, x, 15]
Plot[%, {x, 0, Pi}]
```

Out[∘]=

$$\left(\frac{1}{2}+\frac{2}{\pi}\right)\text{Sin}[x]-\frac{2\,\text{Sin}[2\,x]}{3\,\pi}+\frac{2\,\text{Sin}[3\,x]}{3\,\pi}-\frac{8\,\text{Sin}[4\,x]}{15\,\pi}+\frac{2\,\text{Sin}[5\,x]}{5\,\pi}-\frac{34\,\text{Sin}[6\,x]}{105\,\pi}+\frac{2\,\text{Sin}[7\,x]}{7\,\pi}-\frac{16\,\text{Sin}[8\,x]}{63\,\pi}+\frac{2\,\text{Sin}[9\,x]}{9\,\pi}-\frac{98\,\text{Sin}[10\,x]}{495\,\pi}+\frac{2\,\text{Sin}[11\,x]}{11\,\pi}-\frac{24\,\text{Sin}[12\,x]}{143\,\pi}+\frac{2\,\text{Sin}[13\,x]}{13\,\pi}-\frac{194\,\text{Sin}[14\,x]}{1365\,\pi}+\frac{2\,\text{Sin}[15\,x]}{15\,\pi}$$

Out[∘]=



This is still a relatively poor approximation to the function.

Number 2c.