Homework Set 1 Solutions (9/23 Version)

1. (BH) Consider the differential equation

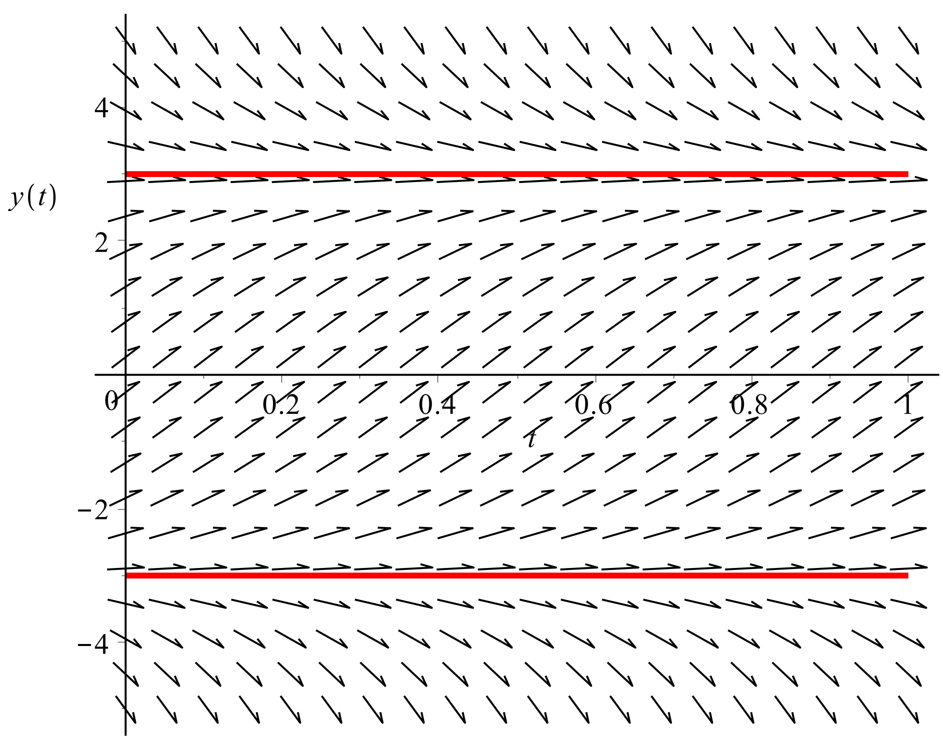
$$\dot{y} + y^2 = 9. \tag{1.1}$$

(a) Find any equilibrium solutions.

Solution. Setting $\dot{y} = 0$, we have $y^2 = 9$, or $y = \pm 3$ as the equilibrium solutions.

(b) Sketch a direction field for (1.1). Indicate the position of the equilibrium solutions.

Solution. Rewriting (1.1), we have $\dot{y} = 9 - y^2$. So if $y < -3$ or $y > 3$, $\dot{y} < 0$. Otherwise, $\dot{y} > 0$. The graph is shown below.



(c) What does your graph tell you will happen to the solution as $t \to \infty$? Be sure to discuss all possible initial conditions.

Solution. If $y(0) > -3$, the arrows show that all these solutions will converge to $y = 3$ as $t \to \infty$. However, if $y(0) < -3$, the arrows show that these solutions will go to $-\infty$ as $t \to \infty$. If $y(0) = \pm 3$, then the solution stays there, since it's an equilibrium solution.

2. (MP) Consider the differential equation

$$\dot{y} + \sin y = 1.$$

Construct a graph showing the direction field and any equilibrium solutions in $t \in [0, 2\pi]$, $y \in [-5, 5]$.

3. Consider the differential equation

$$\dot{y} - 4y = 2e^{-t}, \qquad y(0) = y_0.$$

(a) (BH) Find the solution for any constant y_0.

Solution. Since $p(t) = -4$, the integrating factor is e^{-4t}. Multiplying by this factor and integrating, we have

$$\begin{aligned}
e^{-4t}\dot{y} - 4e^{-4t}y &= 2e^{-5t} \\
\frac{d(e^{-4t}y)}{dt} &= 2e^{-5t} \\
e^{-4t}y &= -\frac{2e^{-5t}}{5} + C \\
y(t) &= -\frac{2e^{-t}}{5} + Ce^{4t} \\
y(0) &= C - \frac{2}{5} = y_0 \\
C &= y_0 + \frac{2}{5} \\
y(t) &= -\frac{2e^{-t}}{5} + \left(y_0 + \frac{2}{5}\right)e^{4t}.
\end{aligned}$$

(b) (BH) Describe how the long-time behavior of y varies with y_0. (In other words, does the solution decay, tend to positive or negative infinity, etc.)

Solution. As $t \to \infty$, $y(t)$ becomes exponentially large, and the sign of $y(t)$ is the same as the sign of $y_0 + 2/5$. Therefore, we have

$$\lim_{t\to\infty} y(t) = \begin{cases} \infty, & y_0 > -2/5, \\ -\infty, & y_0 < -2/5. \end{cases}$$

(c) (BH) Find the critical value of y_0 which separates the two types of behaviors.

Solution. From part (b), we see that the critical value is $y_0 = -2/5$.

(d) (BH) Describe the long-time behavior of y for that specific value of y_0.

Solution. For $y_0 = -2/5$, the solution is $y(t) = -2e^{-t}/5$, which goes to 0 as $t \to \infty$.

(e) (MP) Using the solution you derived in (a), plot integral curves of $y(t)$ for $t \in [0, 0.5]$ and various y_0. Be sure to include the value of y_0 derived in (c).

4. (BH) Show (by deriving the solution, **NOT** by direct substitution) that the solution to the differential equation

$$t\dot{y} + 3(2t+1)y = e^{-6t}, \qquad y(1) = 0$$

is given by

$$y(t) = \frac{e^{-6t}}{3}\left(1 - \frac{1}{t^3}\right).$$

Solution. Dividing by t to obtain the standard form, we have

$$\dot{y} + 3\left(2 + \frac{1}{t}\right)y = \frac{e^{-6t}}{t},$$

so the integrating factor is

$$\mu(t) = \exp\left(\int 3\left(2 + \frac{1}{t}\right) dt\right) = \exp\left(3(2t + \log t)\right) = t^3 e^{6t}.$$

Multiplying and integrating, we have

$$\begin{aligned}
\frac{d}{dt}\left(t^3 e^{6t} y\right) &= t^2 \\
y(t) &= t^{-3}e^{-6t}\left(\frac{t^3}{3} + C\right) = e^{-6t}\left(\frac{1}{3} + \frac{C}{t^3}\right) \\
y(1) &= e^{-6}\left(\frac{1}{3} + C\right) = 0 \\
C &= -\frac{1}{3} \\
y(t) &= \frac{e^{-6t}}{3}\left(1 - \frac{1}{t^3}\right).
\end{aligned}$$

5. (This problem is designed to make you realize that you cannot rely blindly on Mathematica's answers.) Consider the following ODE:

$$\tan\left(\frac{1}{t}\right)\dot{y} + \frac{y}{t^2} = 0.$$

(a) (BH) Calculate the general form for $y(t)$.

Solution. Dividing by the first coefficient to obtain the standard form, we have

$$\dot{y} + \frac{y}{t^2\tan(1/t)} = 0,$$

so by rewriting the reciprocal of the tangent, we see that the integrating factor is

$$\begin{aligned}\mu(t) &= \exp\left(\int \cot\left(\frac{1}{t}\right)\frac{dt}{t^2}\right) = \exp\left(-\int \frac{\cos z}{\sin z}\,dz\right), \quad z = \frac{1}{t}\\ &= \exp(-\log(\sin z)) = \left(\sin\left(\frac{1}{t}\right)\right)^{-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}\left(\left(\sin\left(\frac{1}{t}\right)\right)^{-1} y\right) &= 0\\ y(t) &= C\sin\left(\frac{1}{t}\right).\end{aligned}$$

(b) (MP) Calculate the solution when $y(1/\pi) = 0$ using `DSolve`.
(c) (MP) Calculate the solution when $y(1/\pi) = 0$ using `NDSolve` and plot it for $t \in [-1, 1]$.
(d) (BH) Calculate the solution when $y(1/\pi) = 0$. Do your Mathematica answers miss anything?

Solution.

$$y(1/\pi) = C\sin\pi = 0.$$

Therefore, we see that every solution has $y(1/\pi) = 0$, but Mathematica picks out only one with `NDSolve`.

6. (BH) Let $y = y_1(t)$ be a solution of

$$\dot{y} + p(t)y = 0, \tag{i}$$

and let $y = y_2(t)$ be a solution of

$$\dot{y} + p(t)y = g(t). \tag{ii}$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of (ii).

Solution. Substituting $y = y_1(t) + y_2(t)$ into (ii) and rearranging terms, we have

$$\begin{aligned}\frac{d}{dt}(y_1 + y_2) + p(t)(y_1 + y_2) &= g(t)\\ [\dot{y}_1 + p(t)y_1] + \dot{y}_2 + p(t)y_2 &= g(t).\end{aligned}$$

Since y_1 is a solution of (i), the bracketed expression is zero, which leaves

$$\dot{y}_2 + p(t)y_2 = g(t).$$

But this is just (ii). Since y_2 is a solution of (ii), the result has been proven.

7. (BH) Consider the differential equation

$$t^3\dot{y} + 2t^2 y = t^4 + t^5, \qquad y(1) = y_0.$$

(a) Find the general solution. Where is the solution defined, in general?

Solution. Rewriting in standard form, we have

$$\dot{y} + \frac{2}{t}y = t + t^2,$$

so the integrating factor is

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = \exp(2\log t) = t^2$$

and we have

$$\begin{aligned}
\frac{d(t^2 y)}{dt} &= t^3 + t^4 \\
y(t) &= t^{-2}\left(\frac{t^4}{4} + \frac{t^5}{5} + C\right) = \frac{t^2}{4} + \frac{t^3}{5} + \frac{C}{t^2} \\
y(1) &= \frac{1}{5} + \frac{1}{4} + C = y_0 \\
C &= y_0 - \frac{9}{20}, \\
y(t) &= \frac{t^2}{4} + \frac{t^3}{5} + \frac{1}{t^2}\left(y_0 - \frac{9}{20}\right).
\end{aligned}$$

Therefore, in general the solution is defined for $t \neq 0$.

(b) Are there any particular values of y_0 for which the solution is defined everywhere? If so, calculate them. If not, explain why not.

Solution. The solution will be defined everywhere if the coefficient of t^{-2} is zero, which will occur when $y_0 = 9/20$.

8. (BH) Consider the equation

$$\dot{y} + y^2 = 0, \qquad y(0) = y_0 < 0.$$

(a) Write down the solution to the equation.

Solution. Separating variables, we obtain

$$\begin{aligned}
-\frac{dy}{y^2} &= dt \\
\frac{1}{y} &= t + C \\
y &= (t + C)^{-1} \\
y(0) = y_0 &= C^{-1} \\
y(t) &= \left(t + y_0^{-1}\right)^{-1}.
\end{aligned}$$

(b) How does the interval of existence for the solution depend on y_0?

Solution. The solution exists only for when the quantity in parentheses is positive, so

$$t < -y_0^{-1} = \frac{1}{|y_0|}.$$

9. (BH) Show (by deriving the solution, **NOT** by direct substitution) that a solution of the equation
$$y(t^2-1)\dot{y} = t(y^2-1), \qquad y(0) = 0,$$
is $y = t$. Are there any others? Explain your answer in light of the existence and uniqueness theorem.

Solution. Separating variables, we obtain

$$\begin{aligned}
\frac{y\,dy}{y^2-1} &= \frac{t\,dt}{t^2-1} \\
\frac{\log(y^2-1)}{2} &= \frac{\log(t^2-1)}{2} + C \\
y^2 - 1 &= e^{2C}(t^2-1) \qquad \text{(A)} \\
y(0)^2 - 1 &= e^{2C}(-1) \\
e^{2C} &= 1.
\end{aligned}$$

Using this result in (A) and simplifying, we have the following:

$$\begin{aligned}
y^2 - 1 &= t^2 - 1 \\
y &= \pm\sqrt{t^2} = \pm t.
\end{aligned}$$

There are two solutions to the problem. This is in keeping with the existence and uniqueness theorem since

$$\dot{y} = \frac{t(y^2-1)}{y(t^2-1)},$$

which is not continuous at $y = 0$.

10. Consider the equation
$$\dot{w} = -kt^\alpha w^3, \quad w(1) = 1, \qquad (1.2)$$
where $k > 0$ and α are constants.

(a) (BH) Find the solution of (1.2). Be sure to examine the special case when $\alpha = -1$.

Solution. Separating variables, we obtain

$$\begin{aligned}
-\frac{dw}{w^3} &= kt^\alpha\,dt \\
\frac{1}{2w^2} &= \frac{kt^{\alpha+1}}{\alpha+1} + C
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2w(1)^2} = \frac{1}{2} &= \frac{k}{\alpha+1} + C \\
C &= \frac{1}{2} - \frac{k}{\alpha+1} \\
w^{-2} &= 2\left(\frac{1}{2} + \frac{k(t^{\alpha+1}-1)}{\alpha+1}\right), \\
w(t) &= \left(1 + \frac{2k(t^{\alpha+1}-1)}{\alpha+1}\right)^{-1/2}, \quad \alpha \neq -1, \\
\frac{1}{2w^2} &= k\log t + C, \quad \alpha = -1 \\
\frac{1}{2} &= k\log 1 + C = C \\
w^{-2} &= 2\left(\frac{1}{2} + k\log t\right) \\
w &= (1 + 2k\log t)^{-1/2}, \quad \alpha = -1.
\end{aligned}$$

(b) (MP) Check your answer using Mathematica. Does Mathematica give the solution to every case automatically?

(c) (BH) Discuss the behavior of the solutions to (1.2) as $t \to \infty$. Remark on the solution for all α.

Solution. We have that

$$\begin{aligned}
\lim_{t\to\infty} \frac{kt^{\alpha+1}}{\alpha+1} &= \begin{cases} \infty, & \alpha > -1, \\ 0, & \alpha < -1, \end{cases} \\
\lim_{t\to\infty} k\log t &= \infty, \quad \alpha = -1,
\end{aligned}$$

since k is positive. Therefore, we obtain

$$w(\infty) = \begin{cases} (1+\infty)^{-1/2} = 0, & \alpha \geq -1, \\ \left(1 - \dfrac{2k}{\alpha+1}\right)^{-1/2}, & \alpha < -1. \end{cases}$$

We note that the second line always exists since $\alpha + 1 < 0$, and hence the parenthetical term is always greater than 1.

(d) (MP) Plot integral curves for $k = 3$, $t \in [1, 5]$, and $\alpha = -2, -1, 0, 1, 2$.

In[ ]:= `Quit[]`

HW1 (Checked)

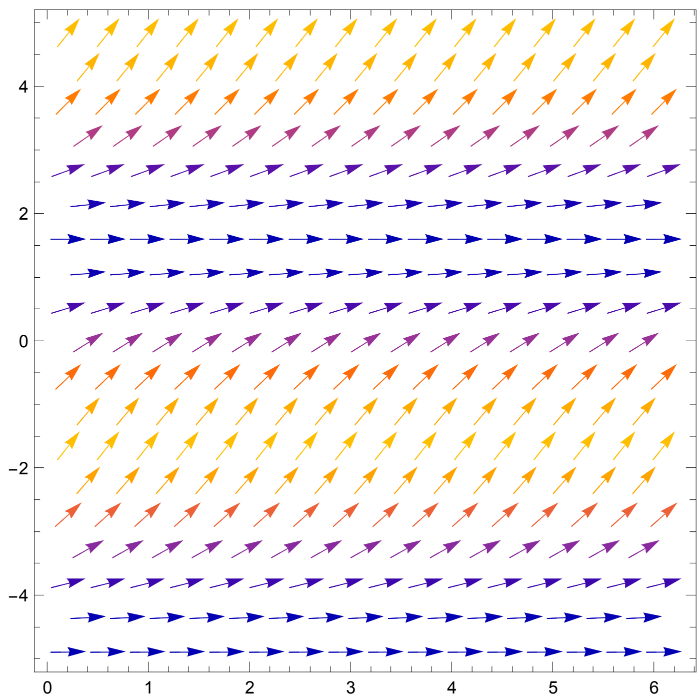
Number 2.

In[ ]:=
```
rhs2 = 1 - Sin[y]
fieldplot = VectorPlot[{1, rhs2}, {t, 0, 2 * Pi}, {y, -5, 5}]
```

Out[ ]=

1 - Sin[y]

Out[ ]=



In[ ]:=
```
Solve[rhs2 == 0, y]
```

Out[ ]=

$$\left\{\left\{y \to \frac{\pi}{2} + 2\pi\, c_1 \text{ if } c_1 \in \mathbb{Z}\right\}\right\}$$

In[ ]:=
```
equil1 = FindRoot[rhs2 == 0, {y, 2}]
equil1a = equil1[[1, 2]]
equil2 = FindRoot[rhs2 == 0, {y, -4}]
equil2a = equil2[[1, 2]]
```

Out[ ]=
```
{y → 1.5708}
```

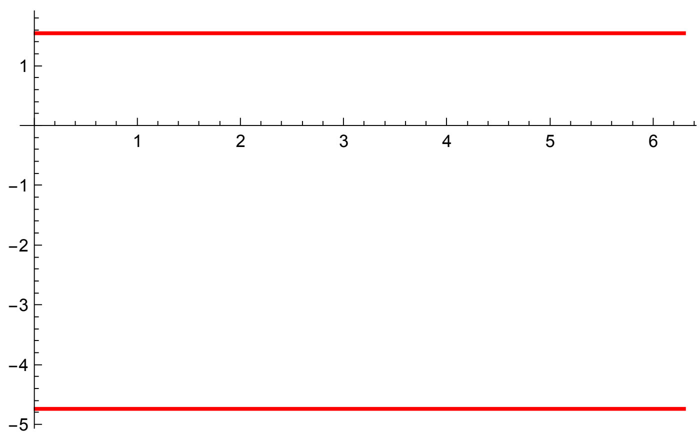
Out[ ]=
```
1.5708
```

Out[ ]=
```
{y → -4.71239}
```

Out[ ]=
```
-4.71239
```

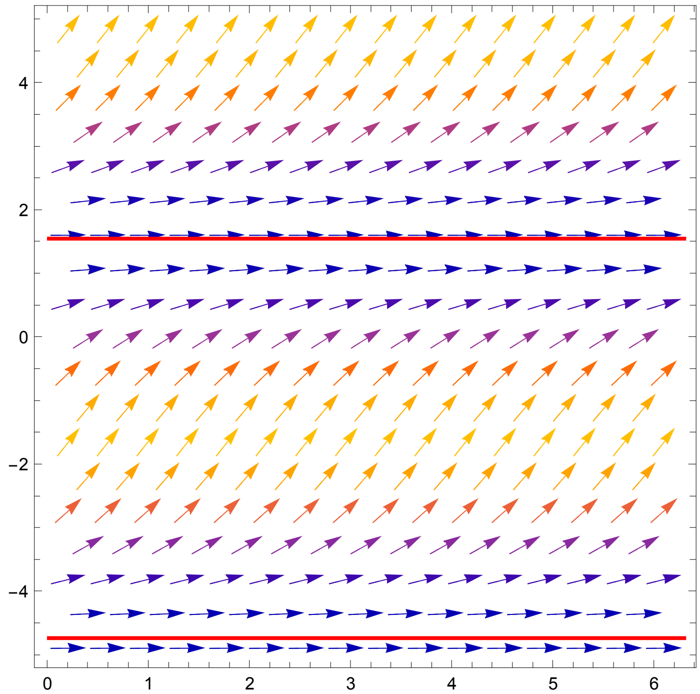
In[ ]:=
```
eqplot = Plot[{equil1a, equil2a}, {t, 0, 2 * Pi}, PlotStyle → {Red, Red}]
```

Out[ ]=



In[ ]:=
```
Show[fieldplot, eqplot]
```

Out[ ]=



Number 3e.

In[ ]:=
```
sol5 = -2 * Exp[-t] / 5 + (y0 + 2 / 5) * Exp[4 * t]
```

Out[ ]=

$$-\frac{2\,e^{-t}}{5} + e^{4\,t}\left(\frac{2}{5} + y0\right)$$

In[ ]:=
```
sol5tab = Table[sol5, {y0, -0.8, 0, 0.2}]
```

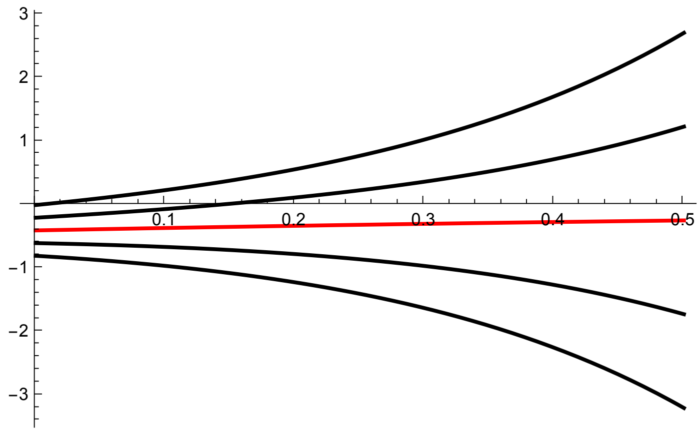
Out[ ]=

$$\left\{-\frac{2\,e^{-t}}{5} - 0.4\,e^{4\,t},\ -\frac{2\,e^{-t}}{5} - 0.2\,e^{4\,t},\ 0. - \frac{2\,e^{-t}}{5},\ -\frac{2\,e^{-t}}{5} + 0.2\,e^{4\,t},\ -\frac{2\,e^{-t}}{5} + 0.4\,e^{4\,t}\right\}$$

Note that the third value in the table is the special one where the solution decays. So we highlight it in the color scheme:

In[∘]:=
```
Plot[sol5tab, {t, 0, 0.5}, PlotRange → All,
 PlotStyle → {Black, Black, Red, Black, Black}]
```

Out[∘]=



Number 5b.

In[∘]:=
```
eq7 = Tan[1 / t] * y'[t] + y[t] / t^2 == 0
numsolve = DSolve[{eq7, y[1 / Pi] == 0}, y[t], t]
```

Out[∘]=

$$\frac{y[t]}{t^2} + \mathrm{Tan}\left[\frac{1}{t}\right] y'[t] == 0$$

Out[∘]=

{{y[t] → 0}}

Number 5c.

In[∘]:=
```
eq7 = Tan[1 / t] * y'[t] + y[t] / t^2 == 0
numsolve = NDSolve[{eq7, y[1 / Pi] == 0}, y[t], {t, -1, 1}]
```

Out[∘]=

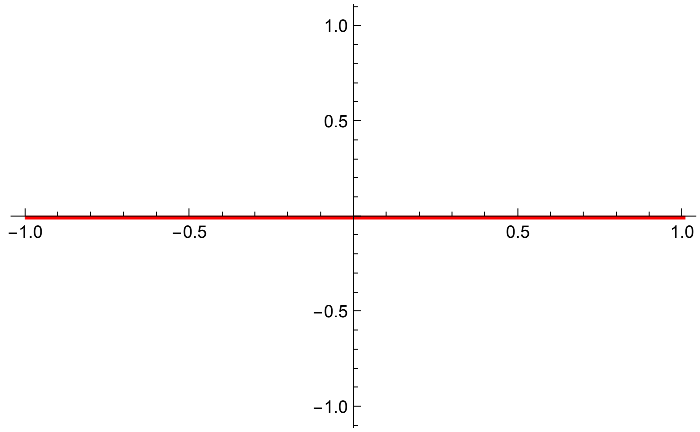
$$\frac{y[t]}{t^2} + \mathrm{Tan}\left[\frac{1}{t}\right] y'[t] == 0$$

Out[∘]=

{{y[t] → InterpolatingFunction[Domain: {{-1., 1.}} Output: scalar][t]}}

In[ ]:=
```
Plot[y[t] /. numsolve, {t, -1, 1}, PlotStyle → {Red}]
```

Out[ ]=



Number 10b.

In[ ]:=
```
eq10 = w'[t] == -k * t ^ (alpha) * (w[t]) ^ 3
sol10 = DSolve[{eq10, w[1] == 1}, w[t], t]
```

Out[ ]=

$$w'[t] == -k\, t^{alpha}\, w[t]^3$$

DSolve: For some branches of the general solution, the given boundary conditions lead to an empty solution.

Out[ ]=

$$\left\{\left\{w[t] \to \frac{\sqrt{1 + alpha}}{\sqrt{1 + alpha - 2\,k + 2\,k\,t^{1+alpha}}}\right\}\right\}$$

Note that the solution doesn't cover the case where α=-1.

Number 10d.

In[ ]:=
```
sol10a = w[t] /. sol10
sol10b = (1 + 2 * k * Log[t]) ^ (-1 / 2)
```

Out[ ]=

$$\left\{\frac{\sqrt{1 + alpha}}{\sqrt{1 + alpha - 2\,k + 2\,k\,t^{1+alpha}}}\right\}$$

Out[ ]=

$$\frac{1}{\sqrt{1 + 2\,k\,Log[t]}}$$

To skip over α=-1, we give the table command a specific list of α values to use.

In[ ]:=
```
sol10tab = Table[sol10a /. (k → 3), {alpha, {-2, 0, 1, 2}}]
```

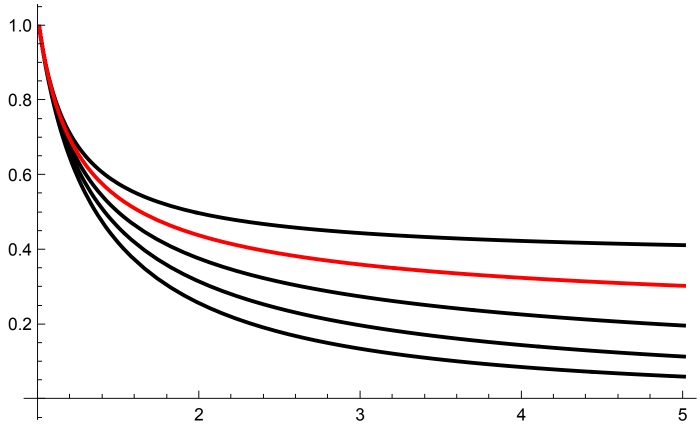
Out[ ]=

$$\left\{\left\{\frac{i}{\sqrt{-7+\frac{6}{t}}}\right\},\left\{\frac{1}{\sqrt{-5+6\,t}}\right\},\left\{\frac{\sqrt{2}}{\sqrt{-4+6\,t^2}}\right\},\left\{\frac{\sqrt{3}}{\sqrt{-3+6\,t^3}}\right\}\right\}$$

Here the special case (with α=-1) is red.

In[ ]:=
```
Plot[{sol10tab, sol10b /. (k → 3)}, {t, 1, 5},
 PlotStyle → {Black, Black, Black, Black, Red}]
```

Out[ ]=



HW2 (Checked)

HW3 (Checked)

HW4 (Checked)

HW5 (Checked)

HW6 (Checked)

HW7 (Checked)

HW8 (Checked)

HW9 (Checked)