Updates

1. The final exam will be administered Sunday, Dec. 15 from 11:30–1:30 **in GOR 204**. You will need to bring a small blue book and your laptop.
2. There will be an informal review session on Friday, Dec. 13 from 10:00-12:00 **in PRN 327**.

Homework Set 8

Read sections 6.2–6.4.

Section 6.2

1. (BH) Use Laplace transforms to find the solution of

$$\ddot{y} + 9y = 1, \qquad y(0) = 0, \quad \dot{y}(0) = 3.$$

2. Consider the following system of differential equations for the three unknowns $\{x(t), y(t), z(t)\}$:

$$\dot{x} + 2x + z = 4, \tag{8.1a}$$
$$\dot{y} + 2y - z = 0, \tag{8.1b}$$
$$\dot{z} = x - y, \tag{8.1c}$$
$$x(0) = 1, \qquad y(0) = 1, \qquad z(0) = 0. \tag{8.2}$$

(a) (BH) Use Laplace transforms to show that $x(t) + y(t) = 2$. (*Hint: Do* **NOT** *attempt to solve for x and y separately.*)
(b) (BH) Use Laplace transforms to show that

$$\hat{x} = \frac{s^2 + 4s + 2}{s(s^2 + 2s + 2)}, \qquad \hat{y} = \frac{s^2 + 2}{s(s^2 + 2s + 2)}.$$

(c) (MP) Invert your answer to part (b) to obtain real solutions for $x(t)$ and $y(t)$. (*Hint: Use the* `ExpToTrig` *function.*) Verify your answer to (a).

3. (BH) Use the appropriate property to show that

$$\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}.$$

4. (MI) (For this problem, you may use Mathematica to do the partial fraction expansion, but do the rest by hand.) Consider the differential equation

$$\ddot{y} - \omega^2 y = e^t, \quad \omega > 0; \qquad y(0) = \dot{y}(0) = 0.$$

(a) Use Laplace transforms to solve the problem for all $\omega \neq 1$.
(b) Use your answer to #3 to show that

$$y(t) = \frac{te^t - \sinh t}{2}, \quad \omega = 1.$$

Section 6.3

5.
(a) (BH) Show that

$$\sum_{n=0}^{\infty} e^{-snT} = \frac{1}{1 - e^{-sT}}. \tag{8.3}$$

(b) (BH) exercise 24

6. (BH) Recall that the *greatest integer function* $\lfloor t \rfloor$ is given by the greatest integer less than or equal to t. Thus $\lfloor 2 \rfloor = 2$, $\lfloor 2.9995 \rfloor = 2$, $\lfloor 3 \rfloor = 3$, etc.
(a) Explain why

$$\lfloor t \rfloor = \sum_{n=1}^{\infty} u_n(t).$$

(b) Show that

$$\mathcal{L}\{\lfloor t \rfloor\} = \frac{1}{s(e^s - 1)}.$$

(*Hint: Use the formula for the sum of a geometric series.*)

7. Consider the function

$$g(t) = (t-2)^2 u_1(t) - (t-3)^2 u_4(t).$$

(a) (MP) Plot $g(t)$ for $t \in [0, 6]$.
(b) (BH) Find $\hat{g}$.
(c) (BH) Find the inverse Laplace transform of

$$\frac{e^{-s}(s-2)}{s^2 + 4}.$$

(d) (BH) Find the inverse Laplace transform of

$$\frac{e^{-(s+2)}}{s^2 - 4}.$$

Section 6.4

8. (BH) Find the solution of

$$\ddot{w} - \dot{w} - 2w = 2 - u_2(t), \qquad w(0) = 1, \quad \dot{w}(0) = -1.$$

9. Consider the problem

$$\ddot{v} + \pi^2 v = f(t), \quad v(0) = 1, \quad \dot{v}(0) = 0; \qquad f(t) = -u_0(t) - 2\sum_{k=1}^{n}(-1)^k u_k(t). \tag{8.4}$$

   (a) (MP) Plot $f(t)$ for $n = 15$, $t \in [0, 40]$.
   (b) (MP) Plot your solution $v(t)$ for $n = 10$, 15, and 20, and $t \in [0, 40]$. Pay careful attention to the amplitude.
   (c) (BH) What do you think happens as $n \to \infty$?

10. In Homework Set 5, #5, we considered the following problem:

$$\ddot{x} + \frac{1}{32}\dot{x} + 96x = F(t), \qquad x(0) = 1/7, \quad \dot{x}(0) = 0, \tag{8.5}$$

   where

$$F(t) = \begin{cases} 4\sin t, & 0 \le t \le 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

   (a) (BH) Write $F(t)$ in unit-step notation.
   (b) (BH) Show that

$$\hat{x} = \left(s^2 + \frac{s}{32} + 96\right)^{-1}\left[4\left(\frac{1 - e^{-2\pi s}}{s^2 + 1}\right) + \frac{1}{7}\left(s + \frac{1}{32}\right)\right]. \tag{8.6}$$

   (c) (MP) Invert (8.6) to determine the solution for $x(t)$.
   (d) (MP) Plot your solution x for $t \in [0, 4\pi]$.