MATH 302-010 | Ordinary Differential Equations
Prof. D. A. Edwards | Due: Nov. 8, 2024

Updates

1. Exam II will be administered on Monday, Oct. 28. You will need to bring a small blue book, as well as your laptop.
2. The exam will cover up through the last homework assignment.

Homework Set 7

Read sections 6.1, 7.6, 7.8, 7.9, 9.3, 9.7.

Section 7.6

1. (BH) Consider the matrix
$$B = \begin{pmatrix} 4 & 3 \\ -6 & -2 \end{pmatrix}.$$
(a) Find the eigenvalues of B.
(b) Classify the fixed point at the origin.
(c) Sketch the phase plane for the system $\dot{\mathbf{x}} = B\mathbf{x}$.

2. (BH) Find the solution of
$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$
You should express your answers in terms of real functions.

Section 7.8

3. Consider the system
$$\dot{\mathbf{x}} = \begin{pmatrix} 6 & -8 \\ 2 & -2 \end{pmatrix}\mathbf{x}. \tag{7.1}$$
(a) (BH) Write the general solution of (7.1)
(b) (MP) Sketch the phase plane for (7.1).

4. (BH) Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{7.2}$$

(a) Write the solution of (7.2) for $\epsilon \neq 0$.
(b) Write the solution of (7.2) for $\epsilon = 0$.
(c) Show that if you take the limit of your answer to (a) for $\epsilon \to 0$, you get (b).

Section 7.9

5. (BH) Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ e^{2t} \end{pmatrix}. \tag{7.3}$$

Using the method of undetermined coefficients, show that a particular solution of this problem is given by

$$\mathbf{x}_{\mathrm{p}} = \frac{1}{3} \begin{pmatrix} 3 - e^{2t} \\ -12 - e^{2t} \end{pmatrix}.$$

Section 9.3

6. Consider the system

$$\dot{x} = xy, \tag{7.4a}$$
$$\dot{y} = y - x^2 + 1. \tag{7.4b}$$

(a) (BH) Find and characterize all the fixed points.
(b) (MP) Sketch the phase plane.

Section 9.7

7. (BH) For the two systems below, characterize the fixed point at the origin, and discuss what might happen to the fixed point if nonlinear terms are added to the equations.

$$\text{(a)}: \quad \begin{aligned} \dot{x} &= -2x - y \\ \dot{y} &= 5x + 2y \end{aligned} \qquad\qquad \text{(b)}: \quad \begin{aligned} \dot{x} &= -4x - y \\ \dot{y} &= x - 2y \end{aligned}$$

8. Consider the system

$$\dot{x} = x(x^2 + \mu) - 3y, \tag{7.5a}$$
$$\dot{y} = x^2 y + 3x. \tag{7.5b}$$

(a) (BH) Show that (7.5) exhibits a Hopf bifurcation as μ passes through zero.
(b) (MP) Sketch one phase plane for the system for μ positive and one for μ negative. Make sure that the axes are large enough to illustrate the limit cycle.

Section 6.1

9. (BH) Calculate the Laplace transform of the following functions. Use the *definition*, not the table. For what range of s will the transforms exist?
   (a) $\cosh \omega t$
   (b) $\sinh \omega t$. In this case,

$$\mathcal{L}\{\sinh \omega t\} = \int_0^\infty e^{-st} \sinh \omega t \, dt. \tag{7.6}$$

   (c) $t \sinh \omega t$. (*Hint: Differentiate (7.6) with respect to s.*)
10. (MP) Calculate either the Laplace transform or the inverse Laplace transform of the following functions:

$$\begin{aligned}
&(a): \qquad \hat{y}(s) = \frac{1}{s^4 - a^4} \\
&(b): \qquad y(t) = t \sin(2\sqrt{t}) \\
&(c): \qquad y(t) = \frac{e^{-1/t}}{\sqrt{t}} \\
&(d): \qquad \hat{y}(s) = \frac{e^{-1/s}}{\sqrt{s}}
\end{aligned}$$