MATH 302-010 — Ordinary Differential Equations
Prof. D. A. Edwards — Due: Oct. 25, 2024

Homework Set 6

Read sections 7.2–7.5.

Section 7.2

1. (BH) Consider the following matrix and vectors:

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}.$$

   (a) Show by direct multiplication that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A, and find the corresponding eigenvalues.
   (b) Consider the three vectors $-\mathbf{v}_1$, $3\mathbf{v}_2$, and $-\mathbf{v}_1 + 2\mathbf{v}_2$. Determine by direct multiplication which (if any) are eigenvectors, and find the corresponding eigenvalues.

2. Consider the following matrix and vector function:

$$B = \begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix}, \qquad \mathbf{w}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-5t}, \qquad \mathbf{w}_2 = t\mathbf{w}_1.$$

   (a) (BH) Show by direct multiplication that $\dot{\mathbf{w}}_1 = B\mathbf{w}_1$.
   (b) (MP) Show by direct multiplication that $\dot{\mathbf{w}}_2 = B\mathbf{w}_2 + \mathbf{w}_1$.

3. Consider the following matrix and vector:

$$C = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

   (a) (BH) Calculate $\det C$.
   (b) (BH) Calculate C^{-1}.
   (c) (BH) Solve $C\mathbf{x} = \mathbf{b}$.
   (d) (MP) Check your answers to (a)–(c) with Mathematica.

Section 7.3

4. (BH) exercise 26

5. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}.$$

   (a) (BH) Calculate the characteristic polynomial of A.
   (b) (BH) Find the eigenvalues of A.
   (c) (BH) Find the eigenvectors of A.
   (d) (MP) Check your answers to (b) and (c) with Mathematica.

Section 7.4

6. (BH) exercise 8(a)–(c)
7. (BH) Consider the vectors

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 6 \\ t \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -2e^t \\ e^t \end{pmatrix}.$$

   (a) Calculate the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.
   (b) Where are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?
   (c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?
   (d) By direct substitution, show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of

$$\dot{\mathbf{x}} = \frac{1}{t+3} \begin{pmatrix} t & -6 \\ (1-t)/2 & 4 \end{pmatrix} \mathbf{x}, \tag{6.1}$$

   and hence verify your answer to (c).

Section 7.5

8. (BH) Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & -1 \\ 2 & -4 \end{pmatrix} \mathbf{x}.$$

   (a) Show that the eigenvalues for this matrix are $\lambda_1 = -2$, $\lambda_2 = -3$.
   (b) Find the general solution $\mathbf{x}(t)$ of this system.
   (c) What happens to the solution as $t \to \infty$?

9. Consider the system
$$\dot{\mathbf{x}} = \begin{pmatrix} -7 & 8 \\ -4 & 5 \end{pmatrix} \mathbf{x}. \tag{6.2}$$

   (a) (BH) Find the solution to (6.2) subject to
$$\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

   (b) (MP) Plot x_1 and x_2 from (a) for $t \in [0, 4]$.

   (c) (BH) For a certain set of vectors $\mathbf{x}_0$, the solution to (6.2) with $\mathbf{x}(0) = \mathbf{x}_0$ decays to zero as $t \to \infty$. Determine $\mathbf{x}_0$.

   (d) (MP) Choose an $\mathbf{x}_0$ which satisfies your answer to (c), then plot x_1 and x_2 for $t \in [0, 4]$.

10. (MP) Consider the system
$$\dot{\mathbf{x}} = \frac{1}{11} \begin{pmatrix} -47 & 2 \\ 12 & -52 \end{pmatrix} \mathbf{x}. \tag{6.3}$$

   (a) Show that the eigenvalues for this system are $\lambda_1 = -4$, $\lambda_2 = -5$, and find the corresponding eigenvectors.

   (b) Find the general solution $\mathbf{x}(t)$ of this system.

   (c) Find the solution of the initial-value problem given by (6.3) and $\mathbf{x}(0) = (4, 0)$.

   (d) Sketch the phase plane for this system.