Homework Set 5 (Revised)

Read sections 3.7, 3.8, 7.1.

Section 3.7

1. A mass is attached to a vertical spring which has internal damping. The relevant equation is given by
$$\ddot{x} + \frac{1}{32}\dot{x} + 96x = 0, \tag{5.1}$$
where x is displacement measured (in cm) from the rest position, and time is measured in seconds. The spring is stretched 1/7 cm and then released.
   (a) (BH) What are the initial conditions that correspond to this situation?
   (b) (MP) Solve the corresponding system.
   (c) (MP) Equipment in the lab can measure the displacement down to a level of 0.01 cm. Estimate the time t_* after which the displacement of the spring *always* remain below the threshold level. (*Hint: Use the amplitude-phase form.*)

2. (BH) The displacement $x(t)$ of a spring is governed by the following equation:
$$9\ddot{x} + 12\dot{x} + 4x = 0, \qquad x(0) = x_0, \quad \dot{x}(0) = v_0.$$
   (a) Construct the solution to this problem.
   (b) Show that $x(t_*) = 0$ if and only if $v_0/x_0 < -2/3$. In this case, how many times does the solution cross the t-axis?

3. In steady state, the temperature $T(x)$ in a domain obeys the following equation:
$$T'' - VT' = 0, \qquad T(0) = 0, \quad T(1) = 1, \tag{5.2}$$
where V is the velocity of heat flow.
   (a) (BH) Solve (5.2) for $V \neq 0$.
   (b) (BH) By taking the limit of your answer to (a), solve (5.2) for $V = 0$.
   (c) (MP) Graph your solution for $V = -2$, 0, and 2. Interpret your solution in terms of the velocity.

Section 3.8

4. (BH) Under a certain model, the amount of income $Y(t)$ in an economy obeys the following equation:

$$\ddot{Y} + 4\dot{Y} + (3+\alpha)Y = -1; \qquad \alpha > 0, \quad \alpha \neq 1. \tag{5.3}$$

   (a) Find the general solution of (5.3), as well as the steady state of Y.
   (b) For what values of α will the solution oscillate?
   (c) Economists would like to maximize the steady state of Y while keeping the income from oscillating. Are those two goals compatible for this model?

5. We reconsider the damped spring of #1, but impose a forcing $F(t)$:

$$\ddot{x} + \frac{1}{32}\dot{x} + 96x = F(t),$$

   where

$$F(t) = \begin{cases} 4\sin t, & 0 \le t \le 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

   The initial conditions are the same as before.
   (a) (MP) Solve the resulting system for the displacement x in the region $t \in [0, 2\pi]$.
   (b) (MP) Show that

$$x(2\pi) \approx 0.0419, \qquad \dot{x}(2\pi) \approx 1.2425.$$

   (c) (BH) Why should x and $\dot{x}$ be continuous at $t = 2\pi$?
   (d) (MP) Using your answer to (c), calculate x in the region $t > 2\pi$.
   (e) (MP) Plot your solution x for $t \in [0, 4\pi]$.

6. (BH) Consider the dimensional equation for the forced spring given in class:

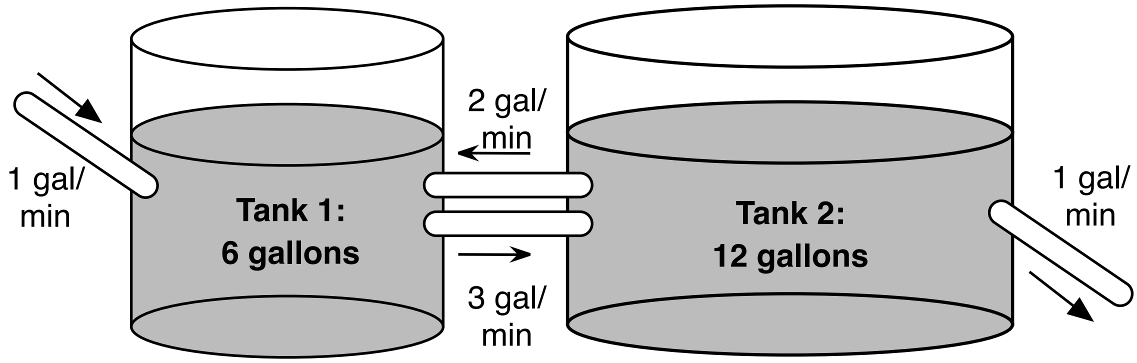
$$M\ddot{x} + b\dot{x} + kx = F(t). \tag{5.4}$$

   If $M = 1$, $b = 1$, $k = 2$, and $F = -\sin t$, find the steady-state solution for the displacement.

7. When scaled in the proper manner, the equation for the velocity $v(r)$ in a cylindrical pipe is given by

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = -a, \quad r \in [0,1]; \qquad v(1) = 0, \tag{5.5}$$

   where $a > 0$ and r is distance from the center.
   (a) (BH) Find the general solution to (5.5). (*Hint: Let $w = dv/dr$.*)
   (b) (BH) Write the solution to the physical system. (*Hint: What is a reasonable condition on v at $r = 0$?*)
   (c) (MP) Plot your solution for $r \in [0, 1]$ and $a = 1, 2, 3, 4$.



Section 7.1

8. Consider the system of two interconnected tanks shown above. Tank 1 contains six gallons of solution and tank 2 contains twelve gallons of a chemical solution (see figure). Fresh water is pumped into tank 1 at the rate of 1 gal/min. Solution from tank 1 to tank 2 is pumped at 3 gal/min. Solution is pumped from tank 2 to tank 1 at 2 gal/min, and solution is removed from tank 2 at 1 gal/min. Let m_i be the mass of the chemical in tank i.
   (a) (BH) Will the volume of solution in the tanks ever change? Why or why not?
   (b) (BH) Show that the system of differential equations governing m_1 and m_2 is given by
   $$6\dot{m}_1 = -3m_1 + m_2, \tag{5.6a}$$
   $$4\dot{m}_2 = 2m_1 - m_2. \tag{5.6b}$$
   (c) (BH) Combine equations (5.6) into a single second-order equation for m_1.
   (d) (MP) Solve your answer to (c) and find expressions for m_1 and m_2 if $m_1(0) = 5$, $m_2(0) = 25$.
   (e) (BH) What happens to m_1 and m_2 as $t \to \infty$? Explain your result physically.
9. (BH) Write the second-order linear ordinary differential equation
   $$(\cos t)\ddot{w} + t^2\dot{w} + e^{-t}w = 0$$
   as a system of two first-order linear ordinary differential equations.
10. (BH) Without memory effects, Newton's Second Law for the displacement $x(t)$ of an object of mass m is given by
    $$m\ddot{x} = F(x, \dot{x}, t). \tag{5.7}$$
    Introduce the momentum p to transform (5.7) into a system of two coupled first-order equations.