MATH 302-010 Ordinary Differential Equations
Prof. D. A. Edwards Due: Oct. 11, 2024

Updates

1. The first exam will be administered on Wednesday, Oct. 2. You will need to bring a small blue book, as well as your laptop.
2. The exam will cover up through section 3.3, so you should do the first two homework problems on this sheet for practice.

Homework Set 4

Read sections 3.3–3.6, 4.2.

Section 3.3/4.2(b)

You may use complex arithmetic as much as you wish to simplify the algebra, but all final expressions should be expressed as real functions.

1. Consider the equation

$$\ddot{x} + 2\dot{x} + 5x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = -(1 + 2\alpha), \quad \alpha \geq 0.$$

   (a) (BH) Construct the solution $x(t)$ in standard form.
   (b) (BH) Convert your answer to (a) into magnitude-phase form.
   (c) (BH) Use your answers to (a) and (b) to confirm (twice) that $x(t_*) = 0$ whenever

$$\tan 2t_* = \frac{1}{\alpha}. \tag{4.1}$$

   (d) (MP) Plot (4.1).
   (e) (BH) By considering your graph in the limits that $\alpha \to 0$ and $\alpha \to \infty$, construct upper and lower bounds on the smallest positive t_*.

2. (BH) Find the general solution of

$$y^{(3)} - 3\ddot{y} + \dot{y} - 3y = 0.$$

Section 3.4

3. (BH) Consider the differential equation

$$\mathcal{L}[y] = a\ddot{y} + b\dot{y} + cy = 0,$$

where the quadratic equation $a\lambda^2 + b\lambda + c = 0$ has the repeated root λ_1.

(a) Show that

$$\mathcal{L}[e^{\lambda t}] = a(\lambda - \lambda_1)^2 e^{\lambda t}. \tag{4.2}$$

Since the right side of (4.2) is zero when $\lambda = \lambda_1$, it follows that $e^{\lambda_1 t}$ is a solution of $\mathcal{L}[y] = 0$.

(b) Show that

$$\frac{\partial}{\partial \lambda}\mathcal{L}[e^{\lambda t}] = 2a(\lambda - \lambda_1)e^{\lambda t} + a(\lambda - \lambda_1)^2 t e^{\lambda t}.$$

Since the right-hand side of the equation is zero when $\lambda = \lambda_1$, conclude that $te^{\lambda_1 t}$ is also a solution of $\mathcal{L}[y] = 0$.

4. Consider the equation

$$y^{(4)} - 8\ddot{y} + 16y = 0. \tag{4.3}$$

(a) (BH) Find the general solution of (4.3).

(b) (MP) Find the solution of (4.3) subject to

$$y(0) = 1, \quad \dot{y}(0) = -3, \quad \ddot{y}(0) = 5, \quad y^{(3)}(0) = -7.$$

Section 3.5

(For the problems in this section, use the method of undetermined coefficients.)

5. (BH) Find the general solution to the differential equation

$$3\ddot{y} + 5\dot{y} - 2y = -2t^2 + 10t. \tag{4.4}$$

6. Consider the differential equation

$$\ddot{y} - \omega^2 y = e^t + e^{-t}. \tag{4.5}$$

(a) (BH) Find the general solution to (4.5) Be sure to account for all $\omega \neq 0$.

(b) (MP) Solve (4.5). Does Mathematica miss anything?

7. Consider the equations

$$6\ddot{y} + 5\dot{y} + y = 20\cos^2\left(\frac{t}{2}\right), \qquad y(0) = 14, \qquad \dot{y}(0) = -1, \tag{4.6a}$$

$$6\ddot{y} + 5\dot{y} + y = 20\cos^4\left(\frac{t}{2}\right), \qquad y(0) = 14, \qquad \dot{y}(0) = -1. \tag{4.6b}$$

   (a) (BH) Find the solution to (4.6a). (*Hint: Use a trigonometric identity to expand the right-hand side.*)
   (b) (MP) Find the solution to (4.6b).
   (c) (MP) Plot the solutions to (4.6a) and (4.6b) on the same graph for $t \in [0, 10\pi]$. Why should the graphs be so similar?

Section 3.6

(For the problems in this section, use the method of variation of parameters.)

8. (BH) Find the general solution to the differential equation

$$\ddot{y} - \omega^2 y = e^t + e^{-t}.$$

   Be sure to account for all $\omega \neq 0$.

9. (BH) Find the general solution of

$$\ddot{y} - 6\dot{y} + 9y = \frac{e^{3t}}{t}.$$

10. Consider the differential equation

$$\ddot{g} + 4g = \sec 2t, \qquad g(0) = 0, \qquad \dot{g}(0) = 0.$$

   (a) (BH) Where is this equation guaranteed to have a unique solution? (*Hint: Don't forget to consider where the initial conditions are given.*)
   (b) (BH) Show that the solution is given by

$$g(t) = \frac{t\sin 2t}{2} + \frac{\log(\cos 2t)\cos(2t)}{4}. \tag{4.7}$$

   Be sure to check the initial conditions.
   (c) (MP) Show that this solution has no extrema for $t > 0$.