Homework Set 3

Read sections 3.2, 4.1, 4.2.

Section 3.2

1. (BH) For the equation
$$\ddot{y} + 5\dot{y} + 6y = 0,$$
find the fundamental set $\{y_1(t), y_2(t)\}$ where
$$y_1(0) = 1, \quad \dot{y}_1(0) = 0; \qquad y_2(0) = 0, \quad \dot{y}_2(0) = 1.$$

2. (BH) Consider the equation
$$(t^2 - 4t + 3)\ddot{y} + 3t\dot{y} + \frac{3y}{\log t} = 0.$$
Find all intervals where this equation is guaranteed to have a unique solution.

3. Consider the equation
$$(t-5)\ddot{y} - t^2\dot{y} - y\sqrt{7-t} = 0, \quad t > 0; \qquad y(0) = 1, \quad \dot{y}(0) = 0. \tag{3.1}$$
   (a) (BH) Find the interval $[a, b)$ where (3.1) is guaranteed to have a unique solution.
   (b) (MP) Plot the solution to (3.1) in $[a, b]$. What happens?
   (c) (MP) Plot the solution to (3.1) in $[a, c]$, where $c > b$. What happens?

4. (BH) Prove that if $\dot{y}_1$ and $\dot{y}_2$ are zero at the same point in I, they cannot be a fundamental set of solutions on that interval.

5. (BH) By considering their Wronskian, show that
$$y_1(\theta) = e^{a\theta}\cos b\theta, \quad y_2(\theta) = e^{a\theta}\sin b\theta, \qquad b \neq 0,$$
are linearly independent for all θ.

6. Consider the equation and boundary condition

$$\frac{d}{dr}\left(r^{1/2}\frac{dy}{dr}\right) = 0, \tag{3.2a}$$

$$y(0) = 1. \tag{3.2b}$$

   (a) (BH) Solve the system (3.2) for y. (You should obtain a one-dimensional family of solutions.)
   (b) (MP) Plot various integral curves of your solution to (a) for $r \in [0, 1]$. What happens near $r = 0$?
   (c) (BH) Does a solution to (3.2a) with the boundary conditions

$$y(0) = 1, \qquad \frac{dy}{dr}(0) = 1$$

   exist? Discuss your answer in light of Theorem 3.2.1 in the textbook.

Section 4.1/4.2(a)

7. (BH) Consider the equation

$$t^2 y^{(4)} + \sqrt{3-t}\,y^{(3)} + \log(t-1)\ddot{y} + \sin t\,\dot{y} + 3y = 0. \tag{3.3}$$

   Find all intervals where (3.3) has a unique solution.

8. Consider the initial-value problem

$$4y^{(3)} + 4\ddot{y} - 3\dot{y} = 0, \qquad y(0) = 3, \quad \dot{y}(0) = -2, \quad \ddot{y}(0) = 2. \tag{3.4}$$

   (a) (BH) Find the solution of (3.4).
   (b) (MP) Plot your solution for $t \in [0, 3]$.

9. (BH) Find the general solution of

$$y^{(4)} - 5\ddot{y} + 4y = 0.$$

10. (MP) Plot the solution to

$$y^{(3)} - (2 + \sin t)y = 0, \quad y(0) = a, \quad \dot{y}(0) = 0, \quad \ddot{y}(0) = a^2,$$

   for $t \in [0, 2\pi]$ and $a = -2, -1, 0, 1,$ and 2.